



**Zonally Averaged Model of Dynamics,  
Chemistry and Radiation for the Atmosphere**

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An important objective of the present joint project with Atmospheric and Environmental Research, Inc., is the development of a *couple* 2-D model of dynamics, radiation and chemistry. Our existing model (see Ko et al. (1985)) is not fully coupled, in the sense that net radiative heating,  $\bar{Q}$ , and the isentropic eddy diffusion coefficient,  $K_{yy}$ , have to be separately specified.

Recent developments (Plumb and Mahlman, 1986; NASA/WMO, 1985) suggest that in the geostrophic limit, the two quantities may be related. It is, however, rather problematic to implement such a geostrophic result in a zonally averaged model because (i) the model is global, while geostrophy breaks down near the equator, and (ii) geostrophy filters out equatorial waves and gravity waves, which are known to be important components in the eddy forcing of the mean flow. Furthermore, the common procedure of adopting the geostrophic result is based on the assumption of conservation of the quasi-geostrophic potential vorticity along isobaric surfaces. (Newman et al., 1986). It is known that a better conserved quantity is the (Ertel's) isentropic potential vorticity along isentropic surfaces.

Our effort during the first six months of the project is to formulate a nongeostrophic theory that is more compatible with our global 2-D model. By relating the Eliassen-Palm flux divergence, which represents the net eddy forcing of the mean zonal momentum, to the flux of the isentropic potential vorticity, we have been able to deduce a nongeostrophic relationship between  $\bar{Q}$  and  $K_{yy}$  in isentropic coordinates. This work is documented in Tung (1986), submitted for publication in *J. Atmos. Sci.*.

We have also been able to infer seasonal and latitudinal distributions of  $K_{yy}$  from the mean momentum budget. The result will be submitted for publication shortly. By adopting a nongeostrophic theory and using Ertel's isentropic potential vorticity in isentropic coordinates instead of quasi-geostrophic potential vorticity in pressure coordinates, many

of the large-scale negative  $K_{yy}$  regions encountered by Newman et al. (1986) disappeared. The remaining large region of negative  $K_{yy}$  occurs in the easterly region of the summer hemisphere, but the magnitudes of  $K_{yy}$  there are extremely small due to the inability of the planetary-stationary waves to significantly penetrate into this easterly region in stratosphere.

During the next six months, an algorithm based on the above-mentioned nongeostrophic relationship will be implemented into the AER 2-D model. This will remove  $K_{yy}$  as an independent input parameter for the model. Self-consistent calculations will be performed for various chemical species, including ozone.

## Bibliography

- Ko, M. K. W., K. K. Tung, D. K. Weisenstein and N. D. Sze, 1985: A zonal mean model of stratospheric tracer transport in isentropic coordinates: numerical simulations for nitrous oxide and nitric acid. *J. Geophys. Res.*, **90**, 2313-2329.
- NASA/WMO, 1985: 1985 Ozone Assessment (draft)
- Newman, P. A., M. R. Schoeberl and R. A. Plumb, 1986: A computation of the horizontal mixing coefficients from observational data, in preparation.
- Plumb, R. A. and J. D. Mahlman, 1986: The zonally-averaged transport characteristics of the GFDL general circulation/tracer model, in preparation.
- Tung, K. K., 1986: Nongeostrophic theory of zonally-averaged circulation, Part I. Formulation, submitted to *J. Atmos. Sci.*.

**Nongeostrophic Theory of Zonally Averaged  
Circulation**

**Part I: Formulation**

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## ABSTRACT

A nongeostrophic theory of zonally averaged circulation is formulated using the nonlinear primitive equations on a sphere, taking advantage of the more direct relationship between the mean meridional circulation and diabatic heating rate which is available in isentropic coordinates. Possible differences between results of nongeostrophic theory and the commonly used geostrophic formulation (e.g. Edmon et al (1980)) are discussed concerning (a) the role of eddy forcing of the diabatic circulation, and (b) the “non-linear nearly inviscid” limit (Held and Hou, 1980) vs the geostrophic limit. Problems associated with the traditional Rossby number scaling in quasi-geostrophic formulations are pointed out and an alternate, more general scaling based on the smallness of mean meridional to zonal velocities for a rotating planet is suggested. Such a scaling recovers the geostrophic balanced wind relationship for the mean zonal flow but reveals that the mean meridional velocity is in general ageostrophic, analogous to the semi-geostrophic theory of Hoskins and Bretherton (1972).

A set of general diagnostic tools comparable in scope to their geostrophic counterparts is given in Part I, including (a) a generalized definition of Eliassen-Palm flux divergence (without restriction to small amplitudes, steady state or to adiabatic flows), the vanishing of which is a necessary condition for nonacceleration (Andrews, 1983), (b) a nonlinear formula that relates the flux of Ertel’s potential vorticity to the Eliassen-Palm flux divergence and (c) a relationship between the Eliassen-Palm flux divergence and isentropic

mixing coefficient,  $K_{yy}$ , used in chemical tracer transport equations in isentropic coordinates. From the mean momentum budget, we give in Part II an estimate of the Eliassen-Palm flux divergence using fitted “observed” field of net radiative heating rate. From this an estimate of the magnitude and latitudinal/seasonal variation of  $K_{yy}$  is also provided.

# 1. Introduction

Great progress has been made in our understanding of the zonally averaged circulation in the atmosphere since the introduction of the concept of residual circulation by Andrews and McIntyre (1976) and Boyd (1976), which helps put in a more proper perspective the role of “eddy” (i.e. deviations from zonal symmetry) in driving the mean (zonally symmetric) circulation. However, our present understanding is based largely on a geostrophic version of the general theory (see Edmon et al (1980), Dunkerton et al (1981), Palmer (1981ab) and Andrews et al (1983)). In its geostrophic form, the set of transformed zonally averaged equations of motion shows clearly that, in the absence of eddies in the form of an Eliassen-Palm flux divergence, eddies do not accelerate the zonal mean flow, a result first recognized by Charney and Drazin (1961) and Dickinson (1969). A more controversial consequence of the geostrophic theory is that at equilibrium, an “almost frictionless” (molecular diffusion only say) stratosphere in the absence of large-scale eddies would be “extremely close” to *radiative equilibrium*, in which the absorption of solar insolation is simply balanced by an increase in local temperature, without inducing a meridional circulation (see Mahlman et al (1984)). It is therefore often concluded (see WMO/NASA (1985), Chapter 7) that global pattern of rising and descending motions in the stratosphere “owes its existence to the presence of asymmetric motions”.

The above cited conclusion appears to have been contradicted by a num-



ber of *zonally symmetric* calculations based on the nonlinear primitive (i.e. nongeostrophic) equations that produce realistic looking Hadley circulations in the absence of large-scale eddies but in the presence of small-scale mixing (i.e. viscosity) (Schneider and Lindzen, 1977; Schneider, 1977; Nakamura, 1978; Held and Hou, 1980). We can further show that even the small-scale eddies are not needed by demonstrating that global scale nonlinear out-of-radiative equilibrium circulations exist in both inviscid and "nearly inviscid" limits, provided that nongeostrophic equations are used.

While these "thought experiments" of a hypothetical axisymmetric atmosphere are useful in highlighting the qualitative difference between the geostrophic and nongeostrophic formulations, they do not address the more practical question: Is our atmosphere in the geostrophic regime? Or is the "nonlinear nearly inviscid" regime (Held and Hou 1980) more applicable? These issues cannot be addressed using a formulation that adopts the *a priori* assumption of geostrophy.

It is important to recognize that there is no *a priori* justification for the geostrophic scaling when applied to the *zonally averaged* zonal momentum equation. When eddy forcing is absent, the geostrophic form of the mean zonal momentum equation suggests that the mean meridional velocity, and so the Coriolis acceleration, vanishes, while geostrophic scaling requires that Coriolis acceleration dominates relative acceleration terms. Therefore, it is obvious that one cannot justifiably use the geostrophic equations to deduce results concerning what happens in the *absence* of large-scale eddies. On

the other hand, the above observation does not necessarily imply that the geostrophic approximation cannot be applied to the stratosphere; given sufficiently strong magnitudes of eddy Eliassen-Palm flux divergence of global length scales, a scaling based on the smallness of the Rossby number may indeed be valid. It is therefore an objective of the present work to (i) point out the different regimes an atmosphere can be in depending on the magnitudes of eddy forcing, and (ii) to find out which of these regimes our atmosphere is in during different seasons. Task (i) is discussed in the present Part I, while task (ii) is relegated to Part II.

Another objective of the present work is to develop a complete set of general diagnostic tools that are comparable in scope to the corresponding geostrophic diagnostics.<sup>1</sup> In order to make our diagnostics more easily adaptable to current procedures for data analysis, we have emphasized the derivation of relationships of various eddy diagnostic quantities to Ertel's potential vorticity (Ertel, 1942). McIntyre and Palmer (1983, 1984) have effectively argued for the usefulness of Ertel's potential vorticity on isentropic surfaces as a diagnostic tool for visualizing large-scale nonlinear dynamical processes in the stratosphere. Hoskins et al (1985) have reviewed the use and significance of isentropic potential vorticity maps in a variety of atmospheric situations, not necessarily restricted to the stratosphere. In particular, they

<sup>1</sup>Such a set of diagnostics for nongeostrophic flow is hitherto not available, although a general (nongeostrophic) definition of the Eliassen-Palm flux divergence in pressure coordinates has been given by Andrews and McIntyre (1976, 1978a) and Boyd (1976) and used by Andrews et al (1983).

emphasized the Lagrangian conservation principle for Ertel's potential vorticity, which presumably is better conserved than that of the quasi-geostrophic potential vorticity along isobaric surfaces (Charney and Stern, 1962). In addition to the two main "principles" associated with the use of isentropic potential vorticity (namely, the conservation principle just mentioned, and the "invertibility" principle (see Hoskins et al (1985)), we add further that (i) the use of isentropic maps of potential vorticity allows us to diagnostically calculate the Eliassen-Palm flux pseudo-divergence, which represents the net eddy forcing of the zonal mean flow in isentropic coordinates, and (ii) the Lagrangian conservative properties of the isentropic potential vorticity endows it with the property of a passive tracer; the dual (dynamical and passive) character of the isentropic potential vorticity allows us, in principle, to deduce the transport characteristics of the atmosphere from its momentum budget.

Although the relationship between the flux of *geostrophic* potential vorticity and *geostrophic* Eliassen-Palm flux divergence is well-known in pressure coordinates (see Edmon et al (1980), and references therein), the corresponding relationship between the flux of Ertel's potential vorticity along isentropic surfaces and the Eliassen-Palm flux divergence does not appear to have been derived.<sup>2</sup> In other words, a set of general diagnostic tools for studying waves and wave-mean flow interactions is not available at the present time except

<sup>2</sup>However, a weak-amplitude limit of such a relationship appears to have been known to Andrews (1983), as can be inferred from his Eq. (4.1).

in the geostrophic limit in pressure coordinates. Partly as a consequence, Clough et al (1985) have to resort to a comparison of the *isentropic* maps of Ertel's potential vorticity with the Eliassen-Palm flux divergence calculated based on the quasi-geostrophic definition of Edmon et al (1980) for *pressure coordinates*. Although this somewhat inconsistent procedure is at present largely dictated by the retrieval process for the observational data, it would have been conceptually clearer if the relationship between the flux of Ertel's potential vorticity along isentropic surfaces and the Eliassen-Palm flux divergence in the same coordinates had been known and a direct (albeit approximate) comparison made of these two quantities. This is especially true for diagnosing model generated data: Although the numerical data are generated using a primitive equation model, the diagnostics by Dunkerton et al (1981) are again based on the quasi-geostrophic formulation of Edmon et al (1980).

A relationship between the flux of Ertel's potential vorticity along isentropic surfaces and the Eliassen-Palm flux divergence is obtained without quasi-geostrophic approximation. This crucial relationship links the flux of a quasi-conservative wave property to the net wave forcing of the mean flow without restriction to small wave amplitudes, thus making it a useful diagnostic tools applicable even to the "surf zones" (McIntyre and Palmer, 1983) in the stratosphere.

The link between the net eddy forcing term in the mean zonal momentum budget and the Ertel's potential vorticity, which is quasi-conservative,

allows us to deduce from the mean momentum budget and mean isentropic gradient of potential vorticity a quantity,  $K_{yy}$ , that appears ubiquitously in chemical tracer transport equations in isentropic coordinates as the isentropic diffusion coefficient (see Tung, 1982; 1984; Ko et al, 1985). Thus it appears that an assessment of the magnitude of the Eliassen-Palm flux divergence could also possibly afford us a preliminary look at the magnitude and latitudinal distribution of  $K_{yy}$ , that has been difficult to obtain by direct evaluation from Eulerian data on transient waves. It also turns out that the use of Ertel's potential vorticity in isentropic coordinates to deduce  $K_{yy}$  is less problem prone than a similar procedure in isobaric coordinates using quasi-geostrophic potential vorticity (Newman et al (1985), personal communication).

The use of isentropic coordinates turns out to be important in our present formulation of a nongeostrophic theory of zonally averaged circulation. Although many of our formulae may have their counterparts in pressure coordinates, the role played by the eddy Eliassen-Palm flux divergence in the forcing of the residual zonal mean circulation becomes more difficult to understand in pressure coordinates due to the presence of mean ageostrophic meridional circulations. The simple, direct relations between the Eliassen-Palm flux divergence and the geostrophic residual mean circulation in pressure coordinates, as discussed succinctly by Edmon et al (1980), are no longer available for nongeostrophic flows. Such a problem does not appear in isentropic co-

ordinates. For our purposes, the advantage of using potential temperature

$$\theta \equiv T \left( \frac{p_{00}}{p} \right)^{R/c_p}$$

as the vertical coordinate (instead of pressure  $p$ , say) lies in the direct relationship between the vertical “velocity”,  $\dot{\theta} \equiv \frac{d}{dt}\theta$ , and the diabatic heating rate,  $Q$ , as given by the thermodynamics equation

$$\dot{\theta} = \frac{\theta}{T} Q$$

Thus there should be no mean circulation in the latitude-height plane for an atmosphere in radiative equilibrium. Conversely the presence of a mean meridional circulation, driven by, say, eddy forcings in the form of Eliassen-Palm flux divergence, necessarily induces a diabatic heating, leading to a non-radiative-equilibrium for the atmosphere. Such a direct relationship between radiation and dynamics is one of the many reasons why it is *convenient* to adopt the isentropic coordinates for our discussions to follow. The relevant equations will also be recast into log-pressure coordinates later in section 7.

## 2. The 2-D equations.

A listing of the set of primitive equations in isentropic coordinates can be found, for example, in Appendix A of Tung (1982). [There is however a typographical error in Eq. (A4) there, where  $\frac{1}{\cos \varphi}$  should read  $\cos \varphi$ ]. The so-called 2-D equations are obtained by taking the zonal average of the 3-D equations by the operation

$$\bar{h}(\varphi, \theta) \equiv \frac{1}{2\pi} \int_0^{2\pi} h(\lambda, \varphi, \theta) d\lambda, \quad (2.1)$$

where  $\lambda$  is the longitude,  $\varphi$  the latitude and  $\theta$  is the potential temperature, used here as the vertical coordinates. We shall use primes to denote deviations from zonal average, i.e.

$$h' \equiv h - \bar{h} \quad (2.2)$$

The primed quantities arise from asymmetric perturbations, and are referred to here as “eddies” or “waves”. In isentropic coordinates it is sometimes more meaningful to use density-weighted zonal averages, defined as (see Gallimore and Johnson (1981 a, b)),

$$\hat{h} \equiv \overline{\rho_\theta h} / \bar{\rho}_\theta, \quad (2.3)$$

where

$$\rho_\theta \equiv \rho \frac{\partial z}{\partial \theta}$$

is the “density” in isentropic coordinates, i.e. it is mass per unit pseudo volume (with  $dz$  replaced by  $d\theta$ ). The deviation from the density-weighted

average is denoted by an asterisk, i.e.

$$h^* \equiv h - \hat{h} \quad (2.4)$$

[Note that  $\overline{\rho_\theta h^*} \equiv 0$ , but  $\overline{h^*} \neq 0$  in general.]

The difference between  $h^*$  and  $h'$  is  $(\overline{\rho'_\theta h'})/\overline{\rho_\theta}$ , which is quadratic in wave amplitudes. The distinction between the two is significant only for finite amplitude waves. But since we are interested in deriving relations that hold in the presence of finite amplitude perturbations, the distinction will be made in the present work.

As in Tung (1982), we use capital letters to denote mass flow rates, thus

$$U \equiv \rho_\theta u, \quad V \equiv \rho_\theta v \cos \varphi, \quad W \equiv \rho_\theta \dot{\theta}. \quad (2.5)$$

With those definitions, the zonally averaged equations take the following form.

## 2.1. Equation of mass conservation.

$$\frac{\partial}{\partial t} \bar{\rho}_\theta + \frac{\partial}{\partial y} \bar{V} + \frac{\partial}{\partial \theta} \bar{W} = 0 \quad (2.6)$$

where  $y \equiv a \sin \varphi$ .

## 2.2. Thermodynamics equation.

The thermodynamics equation

$$\frac{d}{dt} \ln \theta = Q/T$$



can be written as

$$\bar{W} = (\bar{q}/\bar{\Gamma}) \simeq \bar{q}/\bar{\Gamma} \quad (2.7)$$

where  $q \equiv \rho Q$  is the diabatic heating rate per unit physical volume divided by  $c_p$ , and  $\Gamma$  is the static stability parameter defined as

$$\Gamma \equiv \frac{T}{\theta} \frac{\partial \theta}{\partial z}$$

The derivation of (2.7) was previously given by Tung (1982).

[To circumvent any reference to the height coordinate  $z$  in our present formulation in  $\theta$ -coordinate, we note that

$$\Gamma^{-1} = \frac{\theta}{T} \frac{\partial z}{\partial \theta} = \frac{\theta}{\rho T} \rho_\theta = - \frac{\theta}{g \rho T} \frac{\partial}{\partial \theta} p,$$

where the definition  $\rho_\theta \equiv \rho \frac{\partial \rho}{\partial \theta}$  and the hydrostatic relationship

$$g \rho_\theta = - \frac{\partial}{\partial \theta} p$$

have been used. A further use of the ideal gas law;  $p = \rho R T$ , then yields

$$\Gamma^{-1} = - \left( \frac{R}{g} \right) \frac{\partial \ln p}{\partial \ln \theta} = \frac{c_p}{g} \left( 1 - \frac{\partial \ln T}{\partial \ln \theta} \right).$$

This last expression is the one we will use, assuming  $T = T(\lambda, \varphi, \theta)$ . However, in dealing with observational data, we usually have  $\theta = \theta(\lambda, \varphi, p)$ , in which case  $\Gamma$  can be obtained from

$$\Gamma = - \frac{g}{R} \frac{\partial \ln \theta}{\partial \ln p} = \frac{g}{c_p} + \frac{\partial T}{H \partial \ln \left( \frac{p_{00}}{p} \right)},$$

where

$$H = R T / g. \quad ]$$

### 2.3. Zonal momentum equation:

$$\frac{\partial}{\partial t}(\bar{\rho}_\theta \hat{u} \cos \varphi) + \frac{\partial}{\partial y}(\bar{V} \hat{u} \cos \varphi) + \frac{\partial}{\partial \theta}(\bar{W} \hat{u} \cos \varphi) - f\bar{V} = \bar{\nabla} \cdot \bar{F}, \quad (2.8a)$$

where  $f \equiv 2\Omega \sin \varphi$  is the Coriolis parameter. The derivation of Eq. (2.8) is given in Appendix A.

#### a. Eliassen-Palm flux divergence:

Of importance in our present discussion is the right-hand side of Eq. (2.8a), which represents a generalized version of the so-called Eliassen-Palm flux divergence. Written in component form (recalling  $y \equiv a \sin \varphi$ ), it is

$$\bar{\nabla} \cdot \bar{F} = \frac{\partial}{\partial y} F_y + \frac{\partial}{\partial \theta} F_\theta$$

with

$$F_y \equiv -\overline{\rho_\theta u^* v^*} \cos^2 \varphi$$

and

$$F_\theta \equiv -\cos \varphi \left[ \overline{\rho_\theta u^* \dot{\theta}^*} - \frac{1}{g} \overline{p'} \frac{\partial}{\partial x} \Phi' \right] \quad (2.9a)$$

( $\frac{\partial}{\partial x}$  is defined as  $\frac{\partial}{a \cos \varphi \partial \lambda}$ , and  $\Phi$  denotes the Montgomery streamfunction).

A physical interpretation of  $\bar{\nabla} \cdot \bar{F}$  can be given, following Andrews (1983) but generalizing to include diabatic eddies:  $\bar{\nabla} \cdot \bar{F}$  can be viewed as the x-component of the force exerted by eddies on a thin tube with its axis oriented along the x-direction bounded by the fixed lateral sides located at  $y$  and  $y + dy$ , and undulating bottom and top isentropes  $\theta$  and  $\theta + d\theta$ . The

horizontal component of the Eliassen-Palm flux,  $F_v$ , represents lateral flux of zonal momentum into the tube. The first term in the vertical component  $F_\theta$  represents the vertical, i.e. cross isentropic, flux of zonal momentum into the tube, while the second term in  $F_\theta$  represents the x-projection of the net pressure force pushing against the (slanted) isentropes (the so-called form drag).

The use of  $( )^*$  instead of  $( )'$  is necessary in (2.9a) in order for (2.8a) to hold at finite amplitudes. For small amplitude disturbances, the difference between  $( )^*$  and  $( )'$  becomes asymptotically small so that  $\frac{1}{\rho_\theta} F_v \simeq -\overline{u'v'} \cos^2 \varphi$  is the usual horizontal momentum flux by the eddies.

The Eliassen-Palm flux divergence defined in (2.9a) appears to be the most general form possible [It is essentially the same as in Tung (1982), section 5, but with several typographical errors corrected]. Like in Andrews (1983), it is valid without restriction to small amplitude perturbations and quasi-geostrophic scaling. However, unlike in Andrews (1983), where *adiabatic* mean and eddy flows are assumed *a priori*, the present definition is valid for a general diabatic atmosphere. Furthermore, due to his use of the zonally averaged zonal momentum instead of the *density-weighted* zonal average, the eddy flux terms in his zonal momentum equation cannot be expressed in a pure divergence form as is done in our Eq. (2.8a), except under nonacceleration conditions. The present form permits the interpretation that eddy fluxes act to *redistribute* mean angular momentum (without net creation), and that when integrated over a volume bounded by a surface with no net

outward eddy fluxes, eddy forcing of mean flow vanishes.

Nevertheless, the not-in-divergence form of eddy forcing terms of the *zonally averaged* momentum (which cannot be called an Eliassen-Palm flux divergence) is useful in a different aspect. It turns out that an exact relationship exists between this “Eliassen-Palm flux pseudo-divergence”, as we shall call it, and the flux of Ertel’s potential vorticity along isentropic surfaces. Since the flux of Ertel’s potential vorticity is not in general in a divergence form, no exact relationship exists between it and the Eliassen-Palm flux divergence defined in (2.9a).

b. *Eliassen-Palm pseudo-divergence.*

We shall now generalize Andrews’ (1983) expression for the eddy forcing term for the *zonally averaged* zonal momentum to include diabatic terms. Using the zonal average of  $u$  instead of the density-weighted average of  $u$ , we have instead of (2.8a) the following

$$\frac{\partial}{\partial t}(\bar{\rho}_\theta \bar{u} \cos \varphi) + \frac{\partial}{\partial y}(\bar{V} \bar{u} \cos \varphi) + \frac{\partial}{\partial \theta}(\bar{W} \bar{u} \cos \varphi) - f \bar{V} = \square \cdot \mathcal{F}, \quad (2.8b)$$

where  $\frac{1}{\bar{\rho}_\theta} \square \cdot \mathcal{F}$  represents the eddy forcing of the zonally averaged absolute angular momentum, and is given by

$$\begin{aligned} \frac{1}{\bar{\rho}_\theta} \square \cdot \mathcal{F} \equiv & \overline{-v' \cos \varphi \frac{\partial}{\partial y} u' \cos \varphi} - \left( f - \frac{\partial}{\partial y} \bar{u} \cos \varphi \right) \overline{v' \cos \varphi \frac{\rho'_\theta}{\bar{\rho}_\theta}} \\ & - \overline{\dot{\theta}' \frac{\partial}{\partial \theta} u' \cos \varphi} + \frac{1}{\bar{\rho}_\theta} \overline{\rho'_\theta \dot{\theta}'} \frac{\partial}{\partial \theta} \bar{u} \cos \varphi. \end{aligned} \quad (2.9b)$$

By comparing (2.8b) with (2.8a), and noting that

$$\hat{u} - \bar{u} = \overline{\rho'_\theta u' \cos \varphi} / \bar{\rho}_\theta,$$

we see that

$$\begin{aligned}\square \cdot \mathcal{F} &= \bar{\nabla} \cdot \bar{\mathbf{F}} - \frac{\partial}{\partial y} (\bar{V} \overline{\rho'_\theta u'} \cos \varphi / \bar{\rho}_\theta) \\ &\quad - \frac{\partial}{\partial \theta} (\bar{W} \overline{\rho'_\theta u'} \cos \varphi / \bar{\rho}_\theta) - \frac{\partial}{\partial t} \overline{\rho'_\theta u'} \cos \varphi,\end{aligned}\tag{2.10}$$

so that in component form, the “pseudo-divergence” is given by

$$\square \cdot \mathcal{F} = \frac{\partial}{\partial y} \mathcal{F}_y + \frac{\partial}{\partial \theta} \mathcal{F}_\theta + \frac{\partial}{\partial t} \mathcal{F}_t,$$

where

$$\begin{aligned}\mathcal{F}_y &= F_y - \frac{1}{\bar{\rho}_\theta} \bar{V} \overline{\rho'_\theta u'} \cos \varphi = -\overline{V' u'} \cos \varphi \\ \mathcal{F}_\theta &= F_\theta - \frac{1}{\bar{\rho}_\theta} \bar{W} \overline{\rho'_\theta u'} \cos \varphi = -\overline{W' u'} \cos \varphi + \frac{1}{g} \overline{p' \frac{\partial}{\partial \lambda} \Phi'}\end{aligned}$$

and

$$\mathcal{F}_t = -\overline{\rho'_\theta u'} \cos \varphi\tag{2.11}$$

The above expressions reduce to those of Andrews (1983) for adiabatic waves. The presence of the last time-derivative term in (2.10) is what makes  $\square \cdot \mathcal{F}$  not a pure divergence. [The same situation appears also in Andrews’ (1983) expression for adiabatic atmospheres]. In pressure coordinates, there is no distinction between (2.9a) and (2.9b), because the “density”,  $\rho_p$ , in pressure coordinates is  $-\frac{1}{g}$  and so there is no density perturbation, i.e.  $\rho'_p = 0$ . In the present case,  $\square \cdot \mathcal{F}$  can be put into a divergence form only if it is integrated with respect to time over a lifecycle of a conservative disturbance.

**Conservation of absolute angular momentum:**

A statement of the conservation of angular momentum appears to be more obvious if Eq. (2.8a) is combined with Eq. (2.6) to yield

$$\bar{\rho}_\theta \frac{\partial}{\partial t} \hat{L} + \bar{V} \frac{\partial}{\partial y} \hat{L} + \bar{W} \frac{\partial}{\partial \theta} \hat{L} = \bar{\nabla} \cdot \bar{F}, \quad (2.12a)$$

where  $L$  is the absolute angular momentum, with  $L$  defined as

$$L \equiv [u + \Omega a \cos \varphi] \cos \varphi.$$

When  $\bar{L}$  is used instead of  $\hat{L}$ , we have

$$\bar{\rho}_\theta \frac{\partial}{\partial t} \bar{L} + \bar{V} \frac{\partial}{\partial y} \bar{L} + \bar{W} \frac{\partial}{\partial \theta} \bar{L} = \bar{\nabla} \cdot \bar{F} \quad (2.12b)$$

In the following we complete the listing of the zonally-averaged equations for use in our later computations. Some of the additional approximations that will be introduced do not affect our above discussion on Eliassen-Palm fluxes, which are based only on the zonal momentum equation without approximation (beyond that made in deriving the primitive equations).

## 2.4. Meridional momentum equation

It appears, as we will show in section 7, that for a fast rotating planet like the earth, the dominant terms in the mean meridional momentum equation is expressed in the following balance

$$\tilde{f} \bar{u} = -\cos \varphi \frac{\partial}{\partial y} \bar{\Phi} \quad (2.13)$$

where

$$\tilde{f} \equiv \left( 2\Omega + \frac{\bar{u}}{a \cos \varphi} \right) \sin \varphi.$$

and  $\Phi$  is the Montgomery stream function. Eq. (2.13) becomes the geostrophic relationship if  $\tilde{f}$  is approximated by  $f$ .

Within the same degree of approximation as (2.13),  $\bar{u}$  can be replaced by  $\hat{u}$  if (2.8a) is used instead of (2.8b).

## 2.5. Hydrostatic relationship

$$c_p \bar{T} = \theta \frac{\partial}{\partial \theta} \bar{\Phi} \quad (2.14)$$

## 2.6. Balanced wind relationship

Combining (2.13) and (2.14), we find a diagnostic relationship between  $\bar{u}$  and  $\bar{T}$ .

$$2 \left( \Omega + \frac{\bar{u}}{a \cos \varphi} \right) \sin \varphi \theta \frac{\partial}{\partial \theta} \bar{u} = \cos \varphi \frac{\partial}{\partial y} c_p \bar{T}. \quad (2.15)$$

Again,  $\hat{u}$  can replace  $\bar{u}$  in Eq. (2.15).

When the mean relative angular frequency  $\bar{u}/(a \cos \varphi)$  is neglected when compared with the planetary rotational frequency  $\Omega$ , (2.15) becomes the thermal wind relationship

$$f \theta \frac{\partial}{\partial \theta} \bar{u} = \cos \varphi \frac{\partial}{\partial y} c_p \bar{T} \quad (2.16)$$

(2.16) is a good approximation to (2.15) for Earth's atmosphere.

## 2.7. Equation of state

Using the hydrostatic equation

$$g\rho_\theta = -\frac{\partial}{\partial\theta}p \quad (2.17)$$

and the relationship between temperature and pressure from the definition of potential temperature  $\theta \equiv T \left( \frac{p_{00}}{p} \right)^{R/c_p}$ , we can obtain a relationship between density and temperature as

$$\rho_\theta = -\frac{p_{00}}{g} \frac{\partial}{\partial\theta} \left( \frac{T}{\theta} \right)^{c_p/R}. \quad (2.18)$$

To facilitate the taking of zonal averages, temperature is decomposed into a radiative equilibrium state,  $T_e(y, \theta)$ , and deviations from it,  $\Delta T$ , i.e.

$$T \equiv T_e(y, \theta) + \Delta T,$$

and it is assumed that  $\Delta T$  is small compared to  $T_e(y, \theta)$ . (2.18) then yields

$$\bar{\rho}_\theta = -\frac{p_{00}}{g} \frac{\partial}{\partial\theta} \left[ \left( \frac{T_e}{\theta} \right)^{c_p/R} \left( 1 + \frac{c_p}{R} \frac{\Delta \bar{T}}{\bar{T}_e} \right) \right] \quad (2.19)$$

The assumption that  $\Delta T/T_e(y, \theta)$  is small appears to be valid in the stratosphere even for finite amplitude disturbances.

For future reference, we list the expression for the perturbation density

$$\rho'_\theta = -\frac{p_{00}}{g} \frac{\partial}{\partial\theta} \left[ \frac{c_p}{R} \left( \frac{T_e}{\theta} \right)^{c_p/R} \frac{T'}{T_e} \right] \quad (2.20)$$



### 3. Nonacceleration theorem.

The mean angular momentum equation (2.12a) is in the form of the so-called “generalized Eliassen-Palm theorem” (see Andrews and McIntyre (1976, 1978 a,b), Killworth and McIntyre (1985), although we have not shown that our  $-\frac{\partial}{\partial t}\hat{L}$  has the same quadratic property as their  $\frac{\partial}{\partial t}A$ , the rate of change of the pseudo momentum. This is however not important for our purpose of obtaining nonacceleration theorems). If we let

$$\frac{\hat{D}}{Dt} \equiv \frac{\partial}{\partial t} + \hat{v} \frac{\partial}{a \partial \varphi} + \hat{\theta} \frac{\partial}{\partial \theta} \quad (3.1)$$

be the substantial derivative following the density-weighted mean circulation  $(\hat{v}, \hat{\theta}) \equiv (\frac{\bar{v}}{\bar{\rho}_\theta \cos \varphi}, \frac{\bar{w}}{\bar{\rho}_\theta})$ , then Eq. (2.12a) is in the form

$$\bar{\rho}_\theta \frac{\hat{D}}{Dt} \hat{L} - \bar{\nabla} \cdot \bar{\mathbf{F}} = 0. \quad (3.2)$$

[To the right-hand side of Eq. (3.2) one may add a term corresponding to a dissipative force that we have not included in our momentum equation.]

The advantage of using isentropic coordinates becomes clear when we consider the situation under radiative equilibrium conditions. In the absence of diabatic heating, the thermodynamics equation  $\bar{W} = \bar{q}/\bar{\Gamma}$  implies

$$\bar{W} \equiv 0. \quad (3.3)$$

The continuity equation

$$\frac{\partial}{\partial t} \bar{\rho}_\theta + \frac{\partial}{\partial y} \bar{V} + \frac{\partial}{\partial \theta} \bar{W} = 0$$

together with a polar boundary condition

$$\bar{V} = 0 \text{ at } y = a \text{ or } -a.$$

then suggests that

$$\bar{V} \equiv 0 \tag{3.4}$$

at equilibrium. That is, there is no mean meridional circulation at radiative equilibrium. The same conclusion can be drawn in pressure coordinates only if geostrophic approximation is made in the zonal momentum equation. In nongeostrophic form in pressure coordinates, the presence of the advection of mean temperature by ageostrophic mean circulation complicates the relationship between diabatic heating and the mean meridional circulation. Because of the direct relationship that exists in isentropic coordinates between the mean meridional circulation and the presence of nonconservative process (such as diabatic heating), Eq. (3.2) can be written as

$$\bar{\rho}_\theta \frac{\partial}{\partial t} \hat{L} - \bar{\nabla} \cdot \bar{F} = \mathcal{D}, \tag{3.5}$$

where  $\mathcal{D}$  should vanish in the absence of nonconservative processes (please note that a local time derivative is used in (3.5)). Using this so-called generalized Eliassen-Palm relationship, we find that *a necessary condition for nonacceleration in a conservative atmosphere is the vanishing of Eliassen-Palm flux divergence:*

$$\bar{\nabla} \cdot \bar{F} = 0 \tag{3.6}$$

Since

$$\square \cdot \mathcal{F} = \bar{\nabla} \cdot \bar{F}$$

under “nonacceleration conditions”,

$$\nabla \cdot \mathcal{F} = 0 \quad (3.7)$$

is also a necessary condition for nonacceleration.

(3.7) is similar to the condition derived by Andrews (1983) and appears to be the most general version of nonacceleration theorems presently available in Eulerian coordinates. It is valid for finite amplitude waves satisfying primitive equations on a sphere.

It should be pointed out that (3.6) is only a necessary condition for nonacceleration. Vanishing of the Eliassen-Palm flux divergence is in general not sufficient for nonacceleration. Furthermore, although adiabaticity,  $\bar{q} = 0$ , implies  $\bar{\nabla} \cdot \bar{F} = 0$  at steady state, as mentioned above, the converse is not necessarily true. That is, the vanishing of the Eliassen-Palm flux divergence does not guarantee that at steady state the atmosphere should be in local radiative (or radiative-convective) equilibrium, with  $\bar{q} = 0$ . This observation has important implications concerning the presumed role of large-scale waves in driving the atmosphere out of a radiative equilibrium. We will come back to this point in a later section.

#### 4. Relationship between Eliassen-Palm flux pseudo-divergence and the flux of Ertel's potential vorticity.

The relationship between the *quasi-geostrophic* Eliassen-Palm flux divergence in pressure coordinates and the flux of *quasi-geostrophic* potential vorticity is well known (Charney and Stern, 1962; Dickinson, 1969; Green, 1970; Edmon et al, 1980). If the quasi-geostrophic potential vorticity can be assumed to be conserved along isobaric surfaces then a parameterization of the flux, and hence of the Eliassen-Palm flux divergence can be made in terms of the mean gradient of potential vorticity along isobaric surfaces. Such a procedure appears to be problematic because (a) the geostrophic assumption is too restrictive as it cannot account for mixing by gravity waves in the stratosphere, (b) quasi-geostrophic potential vorticity is in general a less-well conserved quantity along isobaric surfaces in the stratosphere than Ertel's potential vorticity along isentropes, and (c) the quasi-geostrophic scaling requires that the cross-isobaric diffusion terms (such as  $K_{yz}$  and  $K_{zz}$ ) be neglected; this is perhaps not justifiable (see Tung (1984)) in pressure coordinates, except in limited regions in the mid stratosphere (see Newman et al (1985), personal communication).

For these reasons it appears to be desirable to obtain a relationship between the nongeostrophic expression we have for the Eliassen-Palm flux

divergence and the flux of Ertel's potential vorticity in isentropic coordinates. Then making use of the quasi-conservative nature of the isentropic potential vorticity, a parameterization of the eddy momentum forcing term (Eliassen-Palm flux divergence) can possibly be made.

#### 4.1. Ertel's potential vorticity

Ertel's potential vorticity is defined as (see e.g. Pedlosky (1979)),

$$\Pi \equiv \frac{\zeta + f}{\rho_\theta} \quad (4.1)$$

where

$$\zeta \equiv \frac{\partial}{\partial x} v - \frac{\partial}{\partial y} u \cos \varphi$$

is the relative vorticity, and

$$\rho_\theta \equiv \rho \frac{\partial z}{\partial \theta} = -\frac{1}{g} \frac{\partial p}{\partial \theta}$$

is the "density" in isentropic coordinates. [Recall also,  $y \equiv a \sin \varphi$ ,  $\frac{\partial}{\partial x} \equiv \frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda}$ ].

It can easily be shown, using (2.4), that the following identity holds for the perturbation (from its density-weighted average ) potential vorticity

$$\rho_\theta \Pi^* = \zeta' - (f - \frac{\partial}{\partial y} \bar{u} \cos \varphi) \rho'_\theta / \bar{\rho}_\theta. \quad (4.2)$$

[Note that although (4.2) is of the same form as the linearized expression for  $\bar{\rho}_\theta \Pi'$ , no small amplitude approximation is made in deriving it]. Using the identity that

$$\overline{a^* b'} = \overline{a' b'},$$

we find that the northward flux of Ertel's potential vorticity is given by

$$\begin{aligned} \overline{\bar{\rho}_\theta (\rho_\theta \Pi^*) v^* \cos \varphi} &= \overline{\bar{\rho}_\theta \bar{\rho}_\theta \Pi^* v'} \cos \varphi \\ &= -\overline{\bar{\rho}_\theta v' \cos \varphi \frac{\partial}{\partial y} u' \cos \varphi} - \left(f - \frac{\partial}{\partial y} \bar{u} \cos \varphi\right) \overline{\rho_\theta' v'} \cos \varphi \end{aligned} \quad (4.3)$$

The expression in (4.3) is to be compared with the Eliassen-Palm flux pseudo-divergence, which is, from (2.9b)

$$\begin{aligned} \square \cdot \mathcal{F} &= -\overline{\bar{\rho}_\theta v' \cos \varphi \frac{\partial}{\partial y} u' \cos \varphi} \\ &\quad - \left(f - \frac{\partial}{\partial y} \bar{u} \cos \varphi\right) \overline{\rho_\theta' v'} \cos \varphi \\ &\quad - \overline{\bar{\rho}_\theta \dot{\theta}' \frac{\partial}{\partial \theta} u' \cos \varphi} + \overline{\rho_\theta' \dot{\theta}' \frac{\partial}{\partial \theta} \bar{u} \cos \varphi} \end{aligned} \quad (4.4)$$

## 4.2. Adiabatic waves

For adiabatic waves, (4.3) and (4.4) are the same, so

$$\square \cdot \mathcal{F} = \overline{\bar{\rho}_\theta (\rho_\theta \Pi^*) v^* \cos \varphi} \quad (4.5)$$

Eq. (4.5) provides an alternate diagnostic expression for the Eliassen-Palm flux pseudo-divergence (the net eddy forcing of the mean flow) which should be valid for time scales less than radiative damping times.. Since no small amplitude assumption has been made, (4.5) should probably hold even for large-amplitude “breaking” planetary waves observed in the stratosphere (McIntyre and Palmer, 1983, 1984; Leovy et al, 1985), to the extent that such waves tend to mix predominantly along isentropes. However, the expression should not be expected to hold in the mesosphere in the presence of significant *breaking* gravity waves, which can mix momentum across isentropes.

### 4.3. Diabatic waves

To give an estimate of the neglected terms in (4.5), we first write down the full expression without approximation,

$$\square \cdot \mathcal{F} = \bar{\rho}_\theta \overline{(\rho_\theta \Pi^*)} v^* \cos \varphi - \overline{\bar{\rho}_\theta \dot{\theta}' \frac{\partial}{\partial \theta} u'} \cos \varphi + \overline{\rho'_\theta \dot{\theta}' \frac{\partial}{\partial \theta} \bar{u}} \cos \varphi. \quad (4.6)$$

The last two terms in (4.6), denoted by  $\Delta$ , will be estimated as follows. Since

$$\bar{\rho}_\theta^2 \left( \frac{1}{\rho_\theta} \frac{\partial}{\partial \theta} u \right)' \simeq \bar{\rho}_\theta \frac{\partial}{\partial \theta} u' - \rho'_\theta \frac{\partial}{\partial \theta} \bar{u},$$

we have

$$\Delta \equiv -\overline{\bar{\rho}_\theta \dot{\theta}' \frac{\partial}{\partial \theta} u'} \cos \varphi + \overline{\rho'_\theta \dot{\theta}' \frac{\partial}{\partial \theta} \bar{u}} \cos \varphi \simeq -\overline{\bar{\rho}_\theta^2 \dot{\theta}' \left( \frac{1}{\rho_\theta} \frac{\partial}{\partial \theta} u \right)'} \cos \varphi$$

The eddy cross-isentropic velocity  $\dot{\theta}'$  is estimated using the Newtonian cooling expression (see Tung (1984))

$$\dot{\theta}' = -\frac{\alpha \theta T'}{\bar{T}}, \quad \text{where } \alpha^{-1} \sim 5 \text{ days}$$

and the vertical derivative of  $u$  is estimated using the thermal wind relationship. Thus

$$\frac{\Delta}{\bar{\rho}_\theta} \simeq -\frac{\alpha}{f_0 \bar{T}} c_p \frac{\partial}{a \partial \varphi} \frac{1}{2} \overline{T'^2}$$

Using a value of  $T' \sim 10^\circ\text{C}$  and a horizontal scale of  $a$ , we find

$$\frac{\Delta}{\bar{\rho}_\theta} \sim 10^{-6} m/s^2$$

This is about two orders of magnitude smaller than the Eliassen-Palm flux divergence, which is of the order of

$$\square \cdot \mathcal{F} / \bar{\rho}_\theta \sim 10^{-4} m/s^2$$

during winter, in the extratropics (Clough et al, 1985).

Thus it appears that the relationship (4.5), which was found for adiabatic waves, should continue to hold for adiabatic waves to a good degree of accuracy. Thus, we have

$$\nabla \cdot \mathcal{F} \simeq \bar{\rho}_\theta \overline{(\rho_\theta \Pi^*)} v^* \cos \varphi \quad (4.7)$$

This is the diagnostic relationship that we have sought as a generalized version of the geostrophic relation commonly used.



## 5. A parameterization of Eliassen-Palm flux pseudo-divergence

To incorporate the effect of diabatic as well as adiabatic eddies, we start with the nonconservative form of Ertel's potential vorticity equation (see Pedlosky (1979)):

$$\rho_\theta \frac{d}{dt} \Pi = S, \quad (5.1)$$

where

$$S \equiv (f + \varsigma) \frac{\partial}{\partial \theta} \dot{\theta}$$

is the diabatic source or sink of potential vorticity. When written in the form of (5.1), the transport of  $\Pi$  obeys the same equation as that for (nonconservative) tracers. It turns out that because of the definition (4.1), the source term is proportional to the Ertel's potential vorticity itself, i.e.

$$S = \rho_\theta \Pi \frac{\partial}{\partial \theta} \dot{\theta} \quad (5.2)$$

To obtain a parameterization of the flux of Ertel's potential vorticity, we take guidance from the small amplitude theory to derive the form of functional dependence on the mean quantity. The perturbation form of (5.1) is

$$\begin{aligned} & \rho_\theta \left( \frac{\partial}{\partial t} + \frac{\hat{v}}{a \cos \varphi} \frac{\partial}{\partial \lambda} \right) \Pi^* + \rho'_\theta \frac{\partial}{\partial t} \hat{\Pi} + \bar{\rho}_\theta \frac{\hat{v}}{a} \frac{\partial}{\partial \varphi} \Pi^* + \bar{\rho}_\theta \hat{\theta} \frac{\partial}{\partial \theta} \Pi^* \\ & + \rho_\theta \frac{\hat{v}^*}{a} \frac{\partial}{\partial \varphi} \hat{\Pi} + \rho_\theta \hat{\theta}^* \frac{\partial}{\partial \theta} \hat{\Pi} = \rho_\theta \Pi^* \frac{\partial}{\partial \theta} \hat{\theta} + \rho_\theta \hat{\Pi} \frac{\partial}{\partial \theta} \dot{\theta}^* \end{aligned} \quad (5.3)$$

(a)
(b)
(c)
(d)
(e)

(f)
(g)
(h)
(i)

As usual, the advection of  $\Pi^*$  by the mean meridional circulation  $(\hat{v}, \hat{\theta})$  (terms (d) and (e)) is to be neglected when compared with the advection by the mean zonal wind (term (b)). Under the same approximation, term (h) should also be neglected as it is of the same order as or smaller than, term (e). An additional assumption is that the time scales of eddies are much shorter than that of the mean quantities, so term (c) should be neglected when compared to term (a), resulting in:

$$\Pi^* = -\eta^* \frac{\partial}{a \partial \varphi} \hat{\Pi} - \phi^* \frac{\partial}{\partial \theta} \hat{\Pi} + \hat{\Pi} \frac{\partial}{\partial \theta} \phi^*, \quad (5.4)$$

where we have defined the displacements  $\eta^*$  and  $\phi^*$  in the horizontal and vertical directions as

$$\begin{aligned} \left( \frac{\partial}{\partial t} + \hat{u} \frac{\partial}{\partial x} \right) \eta^* &= v^* \\ \left( \frac{\partial}{\partial t} + \hat{u} \frac{\partial}{\partial x} \right) \phi^* &= \dot{\theta}^* \end{aligned} \quad (5.5)$$

Thus,

$$\begin{aligned} \overline{(\rho_\theta \Pi^*) v^* \cos \varphi} &= -\overline{v^* (\rho_\theta \eta^*)} \cos^2 \varphi \frac{\partial}{\partial y} \hat{\Pi} \\ &- \overline{v^* \cos \varphi (\rho_\theta \phi^*)} \frac{\partial}{\partial \theta} \hat{\Pi} + \overline{v^* \cos \varphi (\rho_\theta \frac{\partial}{\partial \theta} \phi^*)} \hat{\Pi}. \end{aligned} \quad (5.6)$$

Let

$$\bar{\rho}_\theta K_{vv} \equiv \overline{v^* (\rho_\theta \eta^*)} = \overline{\rho_\theta \frac{\partial}{\partial t} \frac{1}{2} \eta^{*2}} \quad (5.7)$$

$$\bar{\rho}_\theta K_{v\theta} \equiv \overline{v^* (\rho_\theta \phi^*)} \quad (5.8)$$

and

$$\bar{\rho}_\theta \hat{v}_E \equiv \overline{v^* (\rho_\theta \frac{\partial}{\partial \theta} \phi^*)}. \quad (5.9)$$

(5.6) then becomes

$$\begin{aligned} \overline{(\rho_\theta \Pi^*)} v^* \cos \varphi = & -\bar{\rho}_\theta K_{vv} \cos^2 \varphi \frac{\partial}{\partial y} \hat{\Pi} - \bar{\rho}_\theta K_{v\theta} \cos \varphi \frac{\partial}{\partial \theta} \hat{\Pi} \\ & + \bar{\rho}_\theta \hat{v}_E \cos \varphi \hat{\Pi} \end{aligned} \quad (5.10)$$

Thus (4.6) implies that

$$\frac{1}{\bar{\rho}_\theta} \square \cdot \mathcal{F} = -\bar{\rho}_\theta K_{vv} \cos^2 \varphi \frac{\partial}{\partial y} \hat{\Pi} - \bar{\rho}_\theta K_{v\theta} \cos \varphi \frac{\partial}{\partial \theta} \hat{\Pi} + \bar{\rho}_\theta \hat{v}_E \hat{\Pi} \quad (5.11)$$

Since  $\square \cdot \mathcal{F}$  appears as the eddy forcing for the zonal mean absolute angular momentum  $\bar{L}$  (see Eq. (2.12b)), (5.11) should also be rewritten in terms  $\bar{L}$ .

This can be accomplished by noting the following identity

$$\bar{\rho}_\theta \hat{\Pi} = f + \zeta = -\frac{\partial}{\partial y} \bar{L} \quad (5.12)$$

[In particular, the horizontal gradient of the Ertel's potential vorticity is given by

$$\begin{aligned} \bar{\rho}_\theta \frac{\partial}{\partial y} \hat{\Pi} &= -\bar{\rho}_\theta \frac{\partial}{\partial y} \frac{1}{\bar{\rho}_\theta} \frac{\partial}{\partial y} \bar{L} = -\frac{\partial^2}{\partial y^2} \bar{L} + \frac{\frac{\partial}{\partial y} \bar{\rho}_\theta}{\bar{\rho}_\theta} \frac{\partial}{\partial y} \bar{L} \\ &= \frac{2\Omega}{a} - \frac{\partial^2}{\partial y^2} (\bar{u} \cos \varphi) - f \frac{(f - \frac{\partial}{\partial y} \bar{u} \cos \varphi) \Gamma}{\cos \varphi g p_e} \frac{\partial}{\partial \ln \theta} \left[ \frac{p_e}{\tilde{T}_e} \frac{\partial}{\partial \ln \theta} \bar{u} \right], \end{aligned} \quad (5.13)$$

where

$$p_e(\theta) \equiv p_{00} \left( \frac{\tilde{T}_e}{\theta} \right)^{cp/R}$$

Using the parameterization in (5.11), the mean zonal momentum equation becomes

$$\frac{\partial}{\partial t} \bar{L} + (\hat{v} + \hat{v}_E) \cos \varphi \frac{\partial}{\partial y} \bar{L} + \hat{\theta} \frac{\partial}{\partial \theta} \bar{L}$$

$$= K_{yy} \cos^2 \varphi \bar{\rho}_\theta \frac{\partial}{\partial y} \frac{1}{\bar{\rho}_\theta} \frac{\partial}{\partial y} \bar{L} + K_{y\theta} \cos \varphi \bar{\rho}_\theta \frac{\partial}{\partial \theta} \frac{1}{\bar{\rho}_\theta} \frac{\partial}{\partial y} \bar{L}. \quad (5.14)$$

The coefficients  $K_{yy}$  and  $K_{y\theta}$  for the diffusion of mean angular momentum turn out to be the same as those for the diffusion of conservative tracers as described in, for example, Tung (1982; 1984) and Ko et al (1985). Thus it appears that an assessment of the mean angular momentum budget or of the Eliassen-Palm flux divergence could also possibly afford us a preliminary look at the magnitude and latitudinal distribution of the diffusion coefficients used in zonally averaged model of tracer transports. These quantities are difficult to obtain by direct evaluation from data on transient waves in the atmosphere. Before we give a preliminary estimate of  $K_{yy}$  in Part II, a few remarks concerning Eq. (5.14) are given in the following.

- (i) It is well-known that angular momentum  $L$  is not conservative and so cannot be treated in the same way as a conservative tracer. If  $L$  were a conservative quantity, i.e. if it were to satisfy

$$\frac{d}{dt} L = 0,$$

instead of the actual equation

$$\frac{d}{dt} L = - \frac{\partial}{\partial \lambda} \Phi,$$

which we have used, then the procedure outlined in Tung (1982) for conservative tracers would have given, for example, for the first term on the right-hand side of Eq. (5.14),

$$\frac{\partial}{\partial y} K_{yy} \cos^2 \varphi \frac{\partial}{\partial y} \bar{L}$$

instead of

$$K_{\nu\nu} \cos^2 \varphi \bar{\rho}_\theta \frac{\partial}{\partial y} \frac{1}{\bar{\rho}_\theta} \frac{\partial}{\partial y} \bar{L}$$

actually appearing in Eq. (5.14). The difference should be attributed to the nonconservative nature of  $L$ , and not to the “quasi-adiabatic eddy approximation” used to obtain (4.6).

- (ii) If we assume that the vertical to horizontal gradient of the mean Ertel potential vorticity is of the order of the ratio of a scale height to the radius of the earth, then estimates given in Tung (1984) would suggest that the magnitude of the  $K_{\nu\theta}$  term in Eq. (5.14) (or (5.11)) is about 10% of the magnitude of the  $K_{\nu\nu}$  term in that same equation in the *winter* stratosphere in isentropic coordinates [These two terms are comparable in pressure coordinates under the same circumstances]. Therefore the right-hand side of Eq. (5.14) is dominated by the  $K_{\nu\nu}$  term, during winter,

$$\frac{\partial}{\partial t} \bar{L} + (\hat{v} + \hat{v}_E) \cos \varphi \frac{\partial}{\partial y} \bar{L} + \hat{\theta} \frac{\partial}{\partial \theta} \bar{L} = K_{\nu\nu} \cos^2 \varphi \bar{\rho}_\theta \frac{\partial}{\partial y} \frac{1}{\bar{\rho}_\theta} \frac{\partial}{\partial y} \bar{L} \quad (5.15)$$

- (iii) The form of the right-hand side of Eq. (5.14) suggests that large-scale eddies act to (more or less) diffuse mean absolute angular momentum. It is perhaps important to note here the difference between the diffusion of mean absolute angular momentum and the diffusion of mean zonal velocity. The latter mechanism has sometimes been used as a parameterization of the eddy forcing term in the mean zonal momentum equation. It can be shown that the right-hand side of (5.15)

can be written as

$$\begin{aligned}
& K_{yy} \cos^2 \varphi \bar{\rho}_\theta \frac{\partial}{\partial y} \frac{1}{\bar{\rho}_\theta} \frac{\partial}{\partial y} \bar{L} \\
& = K_{yy} \cos^2 \varphi \left[ \frac{\partial^2}{\partial y^2} (\bar{u} \cos \varphi) + \frac{f(f - \frac{\partial}{\partial y} (\bar{u} \cos \varphi)) \Gamma}{\cos \varphi g p_e} \frac{\partial}{\partial \ln \theta} \left( \frac{p_e}{\bar{T}_e} \frac{\partial}{\partial \ln \theta} \bar{u} \right) - \frac{2\Omega}{a} \right]
\end{aligned} \tag{5.16}$$

If the mechanism of zonal velocity diffusion were adopted, the last term in (5.16),

$$- K_{yy} \cos^2 \varphi \frac{2\Omega}{a} \tag{5.17}$$

would have been absent. The presence of this term, which is proportional to the planetary rotational frequency, has important physical implications. This term is negative independent of the sign of mean gradients provided that the flux of Ertel's potential vorticity is down gradient (so that  $K_{yy} > 0$ ). Therefore it provides a source of *easterly* relative momentum. The first two terms in the right-hand side of Eq. (5.26) act to diffuse relative angular momentum, thus serving to smooth horizontal and vertical gradients of the zonal wind  $\bar{u}$ , but providing no net source of mean relative momentum. The presence of (5.17) serves to *decelerate* the westerly jet, balancing the Coriolis acceleration associated with the poleward mean meridional flow that exists in the stratosphere in the winter hemisphere. Without the term (5.17), the stratospheric westerly jet would reach unrealistically large velocities. By thermal wind, the temperature near the winter pole would be too low.

The simple parameterization of Rayleigh friction sometimes used (Leovy,

1964; Holton and Wehrbein, 1980):

$$\frac{1}{\bar{\rho}_\theta} \square \cdot \mathcal{F} = -K_R \bar{u},$$

also provides the needed easterly momentum source in the westerly region. In this sense, Rayleigh friction is probably a better parameterization than the one based on diffusion of  $\bar{u}$ , i.e.

$$\frac{1}{\bar{\rho}_\theta} \square \cdot \mathcal{F} \sim \mu \nabla^2 \bar{u}.$$

(iv) If the same balance of terms as in Eq. (5.15) were to prevail in the easterly region in the summer hemisphere, the presence of the easterly momentum source (5.17) would have led to an unrealistically strong easterly jet. However, the presence of the same easterlies prevents most of the stationary planetary waves from penetrating above 30 km, so that in this region,  $K_{yy}$  should be very small. Also the estimate of the ratio of  $K_{y\theta}$  to  $K_{yy}$  terms mentioned earlier for winter stratosphere probably does not apply to the summer hemisphere. If the eddy forcing term is dominated by the effect of diabatic gravity waves with short vertical scales, the  $K_{y\theta}$  term in Eq. (5.14) may even dominate over the  $K_{yy}$  term in the easterly region. Since  $K_{y\theta}$  can have either sign, the problem of easterly acceleration in a easterly region does not necessarily arise.

(v) Strictly speaking, one should not treat (5.11) or (5.14) as a *parameterization* of the eddy forcing of the zonal mean flow unless the dependence of the coefficients  $K_{yy}$  and  $K_{y\theta}$  on  $\bar{u}$  or  $\bar{L}$  is also known. This

situation for the momentum equation should be contrasted with that for the tracer transport equation, where the  $K$ 's are independent of the concentration of the minor species being transported. Nevertheless, Eq. (5.11) or (5.14) is useful for diagnostic purposes as one can in principle deduce from the momentum budget the same set of  $K$ 's that are used in tracer transport equations.



## 6. Relation between Eliassen-Palm flux divergence and diabatic heating.

The often-cited linear relationship (see WMO/NASA (1985), Chapter 7) between Eliassen-Palm flux term and the diabatic heating is obtained only when the geostrophic approximation is applied to the mean zonal momentum equation, which becomes,<sup>3</sup>

$$-f\bar{V} = \bar{\nabla} \cdot \bar{\mathcal{F}}. \quad (6.1)$$

The steady state continuity equation is, from (2.6),

$$\frac{\partial}{\partial y} \bar{V} + \frac{\partial}{\partial \theta} \bar{W} = 0. \quad (6.2)$$

[Incidentally, it has been shown in Tung (1982) that (6.2) is a good approximation to (2.6) even without the steady-state assumption]. The thermodynamics equation is, in isentropic coordinates, from (2.7):

$$\bar{W} = \bar{q}/\bar{\Gamma} \quad (6.3)$$

Given the eddy forcing,  $\bar{\nabla} \cdot \bar{\mathcal{F}}$ , Eq. (6.1) determines the meridional velocity  $\bar{V}$  locally at every point where the eddy forcing is specified. The vertical velocity  $\bar{W}$  is calculated from  $\bar{V}$  through the continuity equation (6.2), *given a boundary condition on  $\bar{W}$  at some level*. The thermodynamics equation then relates  $\bar{W}$  to the diabatic heating field  $\bar{q}$ .

<sup>3</sup>Note that under the geostrophic approximation,  $\bar{\nabla} \cdot \bar{\mathcal{F}}$  and  $\bar{\nabla} \cdot \bar{F}$  can be used interchangeable in Eq. (6.1)

The physical implications of these relations have often been taken to be that (a) it is the eddies that are responsible for driving the atmosphere away from local radiative equilibrium, and (b) the degree of deviation from radiative equilibrium is directly proportional to the strength of the Eliassen-Palm flux divergence, the last point being inferred from the relationship

$$\frac{\partial}{\partial \theta}(\bar{q}/\bar{\Gamma}) = \frac{\partial}{\partial y} \left( \frac{1}{f} \bar{\square} \cdot \bar{\mathcal{F}} \right), \quad (6.4)$$

obtained by combining (6.1), (6.2) and (6.3).

There appear to be two points of caution that we wish to inject concerning the interpretation just mentioned. First, as mentioned above,  $\bar{W}$  and hence  $\bar{q}$ , are influenced by boundary conditions as well as *in situ* eddy forcing, and so the above results apply only in the absence of significant boundary forcing in  $\bar{W}$ . Second, geostrophy is not a good approximation when applied to the zonally averaged momentum equation in the east-west direction (It is a better approximation when applied to the meridional momentum equation, yielding the thermal wind relationship). Results on zonal mean circulations using the primitive equations are known in some cases to differ significantly from those obtained using geostrophic equations.

This model difference can be understood by examining the relative importance of various terms in the zonal momentum equation:

$$\bar{V} \frac{\partial}{\partial y}(\bar{u} \cos \varphi) + \bar{W} \frac{\partial}{\partial \theta}(\bar{u} \cos \varphi) - f\bar{V} = \bar{\square} \cdot \bar{\mathcal{F}} \quad (6.5)$$

In geostrophic approximation, the nonlinear advection terms, the first two terms in Eq. (6.5), are assumed to be much smaller than the Coriolis term,

the third term in Eq. (6.5). Consequently the balance is between the Coriolis term and the eddy forcing term on the right-hand side (see Eq. (6.1)). As  $\square \cdot \mathcal{F}$  is made smaller and smaller, the Coriolis term necessarily decreases in importance according to the geostrophic balance (6.1). This eventually leads to the breakdown of the geostrophic approximation as the neglected relative acceleration terms are no longer small when compared to the Coriolis force. The unanswered questions are: What is the threshold magnitude for eddy forcing below which the geostrophic balance (6.1) no longer holds, and the “nonlinear nearly inviscid regime” (Held and Hou, 1980) applies? Which regime is our atmosphere in? An assessment of the magnitude of  $\square \cdot \mathcal{F}$  and the validity of geostrophic approximation as applied to the mean zonal momentum equation are necessary for an understanding of these issues. The diagnostic assessments will be given in Part II. In this section, some conceptual issues concerning the state of the atmosphere in the absence of eddy forcing will first be addressed.

The steady state zonal momentum equation can be written as

$$J[\bar{\Psi}, \bar{L}] = \bar{X}, \quad (6.6)$$

where  $\bar{\Psi}$  is the streamfunction of the meridional circulation defined by

$$\bar{W} = \frac{\partial}{\partial y} \bar{\Psi}$$

and

$$\bar{V} = -\frac{\partial}{\partial \theta} \bar{\Psi}. \quad (6.7)$$

The Jacobian operator appearing in (6.6) is defined by

$$J[A, B] \equiv A_y B_\theta - A_\theta B_y.$$

In (6.6)  $\bar{X}$  represents the large-scale eddy forcing plus small-scale dissipative terms, i.e.

$$\bar{X} = \square \cdot \mathcal{F} + \nabla \cdot \mu \nabla \bar{L} \quad (6.8)$$

where  $\mu$  is the coefficient of small-scale mixing or molecular diffusion. In the absence of zonal asymmetries,  $\bar{X}$  is represented by only the (molecular) diffusion.

### 6.1. Inviscid symmetric states.

If the hypothetical atmosphere is symmetric ( $\square \cdot \mathcal{F} = 0$ ) and inviscid ( $\mu \equiv 0$ ), (6.6) becomes

$$J[\bar{\Psi}, \bar{L}] = 0, \quad (6.9)$$

which implies that lines of constant absolute momentum should coincide with the streamline. This is simply a statement that the inviscid symmetric circulation is angular-momentum conserving. There is *no* constraint from the momentum equation that the solution should be in local radiative equilibrium:

$$\bar{\Psi} \equiv 0 \quad (6.10)$$

(which implies, from thermodynamics equation,

$$\bar{q} \equiv 0).$$

This situation should be seen in contrast with the geostrophic version of the zonal momentum equation (6.1), which in the symmetric inviscid case is

$$f\bar{V} = 0,$$

which leads to radiative equilibrium  $\bar{q} \equiv 0$  in the absence of boundary forcing.

The exact solution to (6.9) is

$$\bar{\Psi} = G(\bar{L}); \tag{6.11}$$

the functional form of  $G$  is to be determined by boundary conditions. If the horizontal boundaries are in radiative equilibrium, i.e.

$$\bar{W} = 0 \text{ at } \theta_0 \text{ and } \theta_1, \tag{6.12}$$

we have, without loss of generality

$$\bar{\Psi} = 0 \text{ at } \theta_0 \text{ and } \theta_1. \tag{6.13}$$

If every line of constant  $\bar{L}$  touches either the upper or lower boundary, then (6.11) becomes, when evaluated using (6.13),

$$\bar{\Psi} = G \equiv 0 \tag{6.14}$$

This is the radiative equilibrium solution, with  $\bar{q} \equiv 0$ . The zonal momentum is related through the balanced wind (or thermal wind) relationship to the radiative equilibrium temperature  $T_e$  satisfying  $\bar{q}(T_e) = 0$ .

However, if in the interior of the domain there exists a region of closed contours of  $\bar{L}$ , then there will exist “modon”-like solutions for which  $G$  is not

necessarily zero. There is in general a meridional circulation in such a region (an out-of-radiative equilibrium situation). Also, inside the closed circulation region the zonal momentum is no longer constrained by the radiative equilibrium temperature, and can thus in principle attain any distribution.

The inviscid symmetric circulation is *nonunique* because neither the shape of the closed region, the functional form of  $G$ , nor the distribution of  $\bar{L}$  inside this region is determined without the imposition of additional constraints. A common misconception has been that the radiative equilibrium solution (6.14) is the *only* solution in the absence of eddies or friction. [Ironically, even Held and Hou (1980), who have established that a realistic looking Hadley circulation can exist even in the absence of large-scale eddies, chose to interpret their deduced circulation as due to the presence of small but nonzero viscosity as a singular perturbation to the inviscid solution, which they assumed to be in radiative equilibrium.]

However, that the inviscid steady state recirculating solution can be nonunique is well known in other areas of fluid mechanics (see e.g. Batchelor (1956) in the case of vorticity distribution in steady laminar flow, Stern (1975) for “modons”, and Tung et al (1982) for internal waves of “permanent form”). There are at least two ways of removing the arbitrariness. These depend on whether the steady inviscid solution is treated as the limit of  $\mu \rightarrow 0$  and then  $t \rightarrow \infty$ , or as the limit of  $t \rightarrow \infty$  and then  $\mu \rightarrow 0$ . In the first approach, one considers the “quasi-steady” limit of an inviscid flow (see Benney and Ko (1978) and Tung et al (1982)) and  $G$  can take any arbitrary

form depending presumably on initial conditions (if these are stable). In the second approach, one considers the “nearly inviscid” limit of a steady flow (see Batchelor (1956)). The presence of even a small amount of viscosity (which however acts for an infinitely long time) can place severe constraints on the distribution of  $G$  inside a recirculating region.

## 6.2. The nearly inviscid limit.

The presence of a small amount of viscosity eliminates the radiative equilibrium solution (6.10). More precisely, (6.10), or an order  $\mu$  modification of (6.10), cannot remain as a solution to the viscous equation even in the limit as  $\mu \rightarrow 0$ . The argument was given by Held and Hou (1980) using a version of Hide (1969)’s theorem and assuming that the imposed radiative equilibrium temperature decreases poleward for all heights.

When a small amount of viscosity is present, the inviscid recirculating solution mentioned above it remains valid to leading order except in certain “boundary layers”, in which it may be modified in different ways depending on the boundary condition. If the imposition of viscous boundary conditions, such as no-slip, is inconsistent with the inviscid solution, a (thin) boundary layer is needed. Inside the boundary layer, the effect of viscosity is important and (6.9) no longer holds. Here, the constraint from (6.11) that streamlines follow lines of constant absolute angular momentum is broken. Consequently, the streamlines can be closed, as demanded by mass conservation, while lines of constant angular momentum can be open and intersect the lower boundary,

as required for example by the no-slip condition,  $\bar{u} = 0$  (i.e.  $\bar{L} = \Omega a \cos^2 \varphi$ ).

Outside the boundary layer, the interior solution is almost inviscid, i.e.,

$$\bar{\Psi} \simeq G(\bar{L}) \quad (6.15)$$

except now  $G$  is to be determined by matching to the boundary layer solution, and not by evaluating (6.15) at the boundary itself (where  $\bar{\Psi} = 0$ ).

The detail matched asymptotic solution will be presented in a separate paper. Here we only wish to emphasize the fact that the interior nearly inviscid solution (6.15) can be very close to one of the inviscid recirculating solutions (6.11). And it is therefore more natural to interpret the nearly inviscid solution to be a small perturbation (at least in the interior ) from a recirculating inviscid solution,<sup>4</sup> instead of it being a singular perturbation (with order one change) from the radiative equilibrium solution as suggested by Held and Hou (1980).

The conclusion that one can draw from the above discussion appears to be that eddies (large or small scales) are not “needed” to “drive the atmosphere out of radiative equilibrium”. Nevertheless, sufficient viscosity

may be “needed” to maintain some form of stability for these nearly inviscid

<sup>4</sup>In this case the inviscid solution can be taken as a modon-like recirculating configuration with the lower interface brought arbitrarily close to the lower boundary. The lower interface will in general be a region of discontinuity in which the lines of constant  $\bar{L}$  will complete the closed contour. The addition of a small amount of viscosity will introduce a thin boundary layer around the discontinuity and, depending on the lower boundary condition, lines of constant  $\bar{L}$  do not necessarily have to close via this boundary layer.



steady symmetric circulations.

## 7. A modified scaling appropriate for zonal mean circulations

In section 6, two extreme circulation regimes are discussed. One, the inviscid or nearly inviscid regime, pertains to the mean circulation in the presence of no or small eddy momentum forcing. The other, the geostrophic regime applies away from the equator when eddy momentum forcing is strong. It was clear from the discussion that the geostrophic approximation, obtained using Rossby number scaling, is of only limited validity when applied to different zonally averaged circulation regimes.

In this section, a somewhat different scaling based on the smallness of the ratio of *mean* meridional and zonal velocities is proposed. It turns out that the result is more general and can cover both extremes of circulation regimes mentioned above.

Since the following discussions are not restricted to isentropic coordinates only, the more commonly used equations in pressure coordinates will be used first, with modifications for isentropic coordinates given later.

### 7.1. Implications of Rossby number scaling

Implicit in scalings based on the smallness of Rossby number

$$Ro \equiv \frac{u_{00}/\ell}{2\Omega \sin \varphi_0}, \quad (7.1)$$

where  $u_{00}$  is a typical value of the horizontal velocity,<sup>5</sup>  $\ell$  a typical horizontal

<sup>5</sup>Note that Rossby number scaling does not distinguish between the magnitudes of merid-

length scale of the flow, and  $2\Omega \sin \varphi_0$  a typical (midlatitude) value for the Coriolis parameter  $f$ , is the assumption that the *gradient* of relative momentum be much smaller than the *gradient* of planetary momentum, resulting in the approximation

$$\frac{\bar{D}}{Dt} \cdot \bar{L} \simeq \bar{v} \frac{\partial}{\partial y} (\Omega a \cos^2 \varphi) = -f \bar{v} \quad (7.2)$$

where

$$\frac{\bar{D}}{Dt} \equiv \frac{\partial}{\partial t} + \bar{v} \frac{\partial}{\partial y} + \bar{w} \frac{\partial}{\partial z}, \quad z \equiv H_{00} \ln\left(\frac{p_{00}}{p}\right), \quad w \equiv \frac{d}{dt} z$$

and

$$\bar{L} \equiv \bar{u} \cos \varphi + \Omega a \cos^2 \varphi.$$

(7.2) is the so-called geostrophic approximation, and by its nature necessarily excludes the kind of absolute-angular-momentum-conserving flow cited previously and by Held and Hou (1980), which, by approximately conserving absolute angular momentum,  $\bar{L}$ , generates in its northward branch westerly zonal flows as the planetary angular momentum,  $\Omega a \cos^2 \varphi$ , decreases northward. Therefore, a characteristic of such a flow is that the *variation* (with latitude) of the relative zonal angular momentum is comparable to the *variation* in the planetary angular momentum (even though  $\bar{u}/(\Omega a \cos \varphi)$  is small). In other words, the Rossby number for such a flow should be order one and so cannot be used as an asymptotic parameter for this case.

In our present problem of calculating the zonal mean circulations, the scale of the mean circulation should be obtained as part of the solution instead of specified *a priori*.

ional and zonal velocities.

## 7.2. An alternate scaling

We propose an alternate scaling based on the assumption that for a rapidly rotating planet, such as the Earth, the *zonally averaged* flow should predominately be in the zonal direction. Therefore, the ratio

$$\epsilon \equiv \frac{\bar{v}_{00}}{\bar{u}_{00}} \quad (7.3)$$

should be small. In (7.3),  $\bar{v}_{00}$  and  $\bar{u}_{00}$  are the typical magnitudes of the *zonally averaged* meridional and zonal velocities respectively. [Note that there is no assumption that  $|\frac{v}{u}|$  be small]

If, as is usually done, one nondimensionalizes both horizontal velocities,  $\bar{u}$  and  $\bar{v}$ , by the same  $\bar{u}_{00}$ , then the following scalings should result:

$$\bar{u}_* \equiv \bar{u}/\bar{u}_{00} = \bar{u}_0 + \epsilon \bar{u}_1 + \epsilon^2 \bar{u}_2 + \dots$$

and

$$\bar{v}_* \equiv \bar{v}/\bar{u}_{00} = \epsilon \bar{v}_1 + \epsilon^2 \bar{v}_2 + \dots \quad (7.4)$$

Let  $\ell_y$  and  $\ell_z$  be the typical meridional and vertical length scales, then based on the continuity equation, the mean vertical velocity  $\bar{w}$  should have the scale  $(\frac{\ell_z}{\ell_y} \bar{v}_{00})$ , and so the usual dimensionless quantity

$$\bar{w}_* \equiv \bar{w} \left( \frac{\ell_y}{\ell_z} \right) / \bar{u}_{00}$$

should be of order  $\epsilon$ , i.e.

$$\bar{w}_* = \epsilon \bar{w}_1 + \epsilon^2 \bar{w}_2 + \dots \quad (7.5)$$

For mean transports in the meridional plane, time  $t$  should be scaled by  $\ell_y/\bar{v}_{00}$ , and so we define,

$$t_* \equiv t \left( \frac{\bar{v}_{00}}{\ell_y} \right)$$

Starting with the zonally averaged zonal momentum equation, which we write in the following form

$$\frac{\bar{D}}{Dt} \bar{L} = \bar{X}, \quad (7.6)$$

where  $\bar{X}$  includes the Eliassen-Palm flux divergence (if in pressure coordinates, the residual quantities are used for  $\bar{v}$  and  $\bar{w}$ ), and dividing by  $\bar{u}_{00}^2/\ell_y$ , we have

$$\begin{aligned} & \left[ \varepsilon \frac{\partial}{\partial t_*} + (\varepsilon \bar{v}_1 + \dots) \frac{\partial}{\partial y_*} + (\varepsilon \bar{w}_1 + \dots) \frac{\partial}{\partial z_*} \right] [(\bar{u}_0 + \varepsilon \bar{u}_1 + \dots) \cos \varphi] \\ & - (\varepsilon \bar{v}_1 + \dots) f_* = \varepsilon \bar{X}_1 + \varepsilon^2 \bar{X}_2 + \dots \end{aligned}$$

where

$$y_* \equiv y/\ell_y \text{ and } f_* \equiv f(\ell_y/\bar{u}_{00}). \quad (7.7)$$

In (7.7), we have also defined and scaled

$$\bar{X}_* \equiv \bar{X}/(\bar{u}_{00}^2/\ell_y) = \varepsilon \bar{X}_1 + \varepsilon^2 \bar{X}_2 + \dots \quad (7.8)$$

[The scaling with  $\bar{X}_*$  being order one would have violated our assumption of  $|\bar{v}/\bar{u}|$  small. This exceptional case will not be discussed here]. Therefore, to leading order in  $\varepsilon$ , (7.7) yields

$$\left( \frac{\partial}{\partial t_*} + \bar{v}_1 \frac{\partial}{\partial y_*} + \bar{w}_1 \frac{\partial}{\partial z_*} \right) (\bar{u}_0 \cos \varphi) - \bar{v}_1 f_* = \bar{X}_1 \quad (7.9)$$

Note that both the relative acceleration and Coriolis force are retained in this approximation. In the  $y$ -direction, we start with the following zonally averaged meridional momentum equation

$$\frac{\bar{D}}{Dt} \bar{v} + \tilde{f} \bar{u} = -\frac{1}{a} \frac{\partial}{\partial \varphi} \bar{\Phi} + \bar{Y}, \quad (7.10)$$

where  $\bar{Y}$  is the eddy forcing for the  $y$ -momentum and  $\tilde{f} \equiv (2\Omega + \frac{a}{a \cos \varphi}) \sin \varphi$ .

The same procedure as before yields

$$\begin{aligned} & \left[ \varepsilon \frac{\partial}{\partial t^*} + (\varepsilon \bar{v}_1 + \dots) \frac{\partial}{\partial y_*} + (\varepsilon \bar{w}_1 + \dots) \frac{\partial}{\partial z_*} \right] [\varepsilon \bar{v}_1 + \dots] \\ & + (\tilde{f}_0 + \varepsilon \tilde{f}_1 + \dots) (\bar{u}_0 + \varepsilon \bar{u}_1 + \dots) \\ & = -\cos \varphi \frac{\partial}{\partial y_*} (\bar{\Phi}_0 + \varepsilon \bar{\Phi}_1 + \dots) + (\varepsilon \bar{Y}_1 + \varepsilon^2 Y_2 + \dots), \end{aligned} \quad (7.11)$$

where

$$\begin{aligned} \tilde{f}_0 &= f_* + \bar{u}_0 \left( \frac{\ell_y}{a} \right) \tan \varphi \\ \tilde{f}_1 &= \bar{u}_1 \left( \frac{\ell_y}{a} \right) \tan \varphi. \end{aligned}$$

To leading order in  $\varepsilon$ , (7.11) is

$$\tilde{f}_0 \bar{u}_0 = -\cos \varphi \frac{\partial}{\partial y_*} \bar{\Phi}_0, \quad (7.12a)$$

which is the balanced wind relationship. Modification to (7.12) from the mean meridional advection terms arises at order  $\varepsilon^2$ . Thus, the balanced wind relations (7.12) holds to high accuracy provided the eddy meridional momentum forcing is not too strong (i.e. if  $\bar{Y}_1 = 0$ ).

Eq. (7.12) reduces to the geostrophic relation if one further assume that the Rossby number based on the radius of the earth  $a$ ,

$$R_0|_a \equiv \frac{\bar{u}/\cos\varphi}{2\Omega a} \quad (7.13)$$

is small. This is generally true for a rotating planet whose planetary rotational frequency,  $\Omega$ , is larger than the relative angular frequency,  $\bar{u}/(a \cos \varphi)$ , of the flow on it. [Note that this ratio does not involve the unknown length scale  $\ell_y$ .] Thus a special, relevant case of (7.12) is

$$f_* \bar{u}_0 = -\cos\varphi \frac{\partial}{\partial y_*} \bar{\Phi}_0 \quad (7.14a)$$

This geostrophic relationship for the zonal flow is obtained *without* the Rossby number scaling, i.e. the smallness of the Rossby number defined in (7.1) is *not assumed*.

Assuming  $\bar{Y}_1 \equiv 0$ , the next order in (7.11) is

$$(f_* + 2\bar{u}_0 \left( \frac{\ell_y}{a} \tan \varphi \right)) \bar{u}_1 = -\cos\varphi \frac{\partial}{\partial y_*} \bar{\Phi}_1, \quad (7.12b)$$

and if furthermore (7.13) is small, (7.12b) becomes again the geostrophic relation:

$$f_* \bar{u}_1 = -\cos\varphi \frac{\partial}{\partial y_*} \bar{\Phi}_1 \quad (7.14b)$$

The energy equation,

$$\frac{d}{dt} \ell n \theta = \frac{Q}{T},$$

when rewritten in log-pressure coordinates, becomes

$$\frac{d}{dt} T + \frac{g}{c_p} \frac{T}{T_{00}} w = Q,$$

where  $T_{00}$  is the reference temperature associated with the reference scale height,  $H_{00} \equiv RT_{00}/g$ , used in the definition of  $z$ . The zonal mean of the above equation is

$$\frac{\bar{D}}{Dt}\bar{T} + \bar{w}\frac{\bar{T}}{T_{00}}\frac{g}{c_p} = \bar{Q} + \bar{Z}, \quad (7.15)$$

where  $\bar{Z}$  contains the eddy heat fluxes. We shall assume that  $(\bar{v}, \bar{w})$  is the residual mean circulation, so that  $\bar{Z}$  is not of dominant importance in the energy equation. This assumption will be reflected in our scaling for  $\bar{Z}$  to follow.

We choose

$$\begin{aligned} \bar{T}_* &\equiv \bar{T}/T_{00} = \bar{T}_0 + \varepsilon\bar{T}_1 + \dots \\ \bar{Z}_* &\equiv \bar{Z}\ell_y/(\bar{u}_{00}T_{00}) = \varepsilon\bar{Z}_1 + \varepsilon^2\bar{Z}_2 + \dots \\ \text{and } \bar{Q}_* &\equiv \bar{Q}\ell_y/(\bar{u}_{00}T_{00}) = \bar{Q}_0 + \varepsilon\bar{Q}_1 + \dots \end{aligned} \quad (7.16)$$

For our present purpose, we shall assume that the mean radiative heating rate  $\bar{Q}$  is a function of the mean temperature, viz,

$$\bar{Q} = \bar{Q}(\bar{T}).$$

Therefore,

$$\bar{Q}_* = \bar{Q}_*(\bar{T}_*) = \bar{Q}_*(\bar{T}_0) + \varepsilon\frac{\partial}{\partial T_0}\bar{Q}_*(\bar{T}_0)\bar{T}_1 + \dots \quad (7.17)$$

and thus

$$\bar{Q}_0 = \bar{Q}_*(\bar{T}_0) \quad (7.18a)$$

$$\bar{Q}_1 = \frac{\partial}{\partial T_0}\bar{Q}_*(\bar{T}_0) \cdot \bar{T}_1, \text{ etc.} \quad (7.18b)$$



Eq. (7.15), when divided by  $(\bar{u}_{00}T_{00})/\ell y$ , becomes

$$\begin{aligned}
& \left[ \varepsilon \frac{\partial}{\partial t_*} + (\varepsilon \bar{v}_1 + \dots) \frac{\partial}{\partial y_*} + (\varepsilon \bar{w}_1 + \dots) \frac{\partial}{\partial z_*} \right] [\bar{T}_0 + \varepsilon \bar{T}_1 + \dots] \\
& + (\varepsilon \bar{w}_1 + \dots) (\bar{T}_0 + \varepsilon \bar{T}_1 + \dots) \left( \frac{\ell z}{T_{00}} \frac{g}{c_p} \right) \\
& = \bar{Q}_0 + \varepsilon \bar{Q}_1 + \dots + \varepsilon \bar{Z}_1 + \varepsilon^2 \bar{Z}_2 + \dots
\end{aligned} \tag{7.19}$$

The leading term in (7.18) is

$$\bar{Q}_0 \equiv 0, \tag{7.20}$$

which defines the leading order temperature field via (7.18a) as

$$\bar{T}_0 = T_e, \tag{7.21a}$$

where

$$\bar{Q}_*(T_e) \equiv 0, \tag{7.21b}$$

In other words,  $\bar{T}_0$  is the radiative equilibrium temperature  $T_e$  satisfying (7.21b). At the next order in  $\varepsilon$ , (7.19) yields

$$\begin{aligned}
& \left( \frac{\partial}{\partial t_*} + \bar{v}_1 \frac{\partial}{\partial y_*} + \bar{w}_1 \frac{\partial}{\partial z_*} \right) \bar{T}_0 + \bar{w}_1 \bar{T}_0 \left( \frac{\ell z}{T_{00}} \frac{g}{c_p} \right) \\
& = \bar{Q}_1 + \bar{Z}_1,
\end{aligned} \tag{7.22}$$

which together with (7.18b), should determine  $\bar{T}_1$ , the mean temperature change (from radiative equilibrium) induced by the mean meridional circulation  $(\bar{v}_1, \bar{w}_1)$ .

### 7.3. Summary of scaled mean equations in isobaric coordinates

In dimensional form, we write the zonal mean temperature and zonal flow as

$$\bar{T} = T_e(y, z) + \Delta \bar{T} \quad (7.23)$$

$$\bar{u} = u_e(y, z) + \Delta \bar{u} \quad (7.24)$$

where  $T_e$  is the local radiative equilibrium temperature determined from

$$\bar{Q}(T_e) \equiv 0 \quad (7.25)$$

And  $u_e$  is related to  $T_e$  through the balanced wind relationship

$$2 \left( \Omega + \frac{\bar{u}}{a \cos \varphi} \right) \sin \varphi \frac{\partial}{\partial z} \bar{u} = \cos \varphi_0 \frac{R}{H_{00}} \bar{T} \quad (7.26)$$

(Substitute  $u_e$  for  $\bar{u}$  and  $T_e$  for  $\bar{T}$  in (7.26)) (7.26) is obtained by combining the hydrostatic relation

$$H_{00} \frac{\partial}{\partial z} \bar{\Phi} = R \bar{T}, \quad (7.27)$$

with the balanced equation

$$\left( 2\Omega + \frac{\bar{u}}{a \cos \varphi} \right) \sin \varphi \cdot \bar{u} = -\cos \varphi \frac{\partial}{\partial y} \bar{\Phi}, \quad (7.28)$$

which is correct up to and including order  $\varepsilon$ .

The zonal momentum equation is

$$\left( \frac{\partial}{\partial t} + \bar{v} \frac{\partial}{\partial y} + \bar{w} \frac{\partial}{\partial z} \right) u_e \cos \varphi - f \bar{v} = \bar{X} \quad (7.29)$$

Knowing  $u_e$  from (7.25) and (7.26), this is a linear equation for the meridional circulation  $(\bar{v}, \bar{w})$  assuming that the eddy momentum forcing,  $\bar{X}$ , is known.

$\bar{v}$  and  $\bar{w}$  are related to each other by the continuity equation, which is

$$\frac{\partial}{\partial y}(\bar{v} \cos \varphi) + \frac{1}{\rho_0 z} \frac{\partial}{\partial z}(\rho_0(z) \bar{w}) = 0, \quad (7.30)$$

where  $\rho_0(z) \equiv \rho_0(0)e^{-z/H_{00}}$ .

Knowing  $\bar{v}$  and  $\bar{w}$ , the energy equation

$$\begin{aligned} \left( \frac{\partial}{\partial t} + \bar{v} \frac{\partial}{\partial y} + \bar{w} \frac{\partial}{\partial z} \right) T_e + \bar{w}(T_e/T_{00}) \frac{g}{c_p} = \bar{Z} \\ + \frac{\partial}{\partial T} \bar{Q}(T_e) \cdot \Delta \bar{T} \end{aligned} \quad (7.31)$$

then yields the dynamically induced part of the mean temperature  $\Delta \bar{T}$ . And from  $\Delta \bar{T}$ , the balanced wind relation gives diagnostically the dynamical correction to  $u_e$ . Note that the solution procedure outlined above for the scaled equations is linear in every step, while the original set of mean equations is nonlinear.

Note also that the above set of equations includes the geostrophic equations as a subset; one can make the additional assumption of small Rossby numbers,  $R_0 \ll 1$ , and the geostrophic equations will result from the above set. It also includes the nearly inviscid regime mentioned in section 6; even for small or zero eddy forcing  $\bar{X}$ , a mean meridional circulation  $(\bar{v}, \bar{w})$  and an out of equilibrium temperature distribution  $T_e + \Delta \bar{T}$  can still be the solution of our scaled set of equations, given appropriate boundary conditions.

## 7.4. The scaled mean equations in isentropic coordinates

The set of scaled mean equations in isentropic coordinates is the same as stated in section 2, except

- (i) with the radiative equilibrium temperature  $T_e(y, \theta)$  replacing  $\bar{T}$  in the static stability parameter appearing in the thermodynamics equation (2.7).
- (ii) with  $u_e$  replacing  $\hat{u}$  in zonal the momentum equation (2.8a) and  $\bar{u}$  in (2.8b), and similarly in (2.12a) and (2.12b).
- (iii) with  $\bar{q}(\bar{T})$  approximated by  $\bar{q} = \frac{\partial}{\partial T} \bar{q}(T_e) \cdot (\bar{T} - T_e)$ . and
- (iv) under the present scaling, there is no difference between  $\bar{\nabla} \cdot \bar{F}$  and  $\square \cdot \mathcal{F}$ .

## 8. Conclusions

A formulation is given for a nongeostrophic theory of zonally averaged circulation. Motivations for extending the commonly used geostrophic version of the general theory are many; some of them are

- (a) Geostrophy does not hold at and near the equator and so is not uniformly valid in a model of zonal mean global circulation.
- (b) Although there is some consensus that the mean zonal velocity is in geostrophic balance, there is no *a priori* justification for applying the geostrophic approximation for the zonally averaged north-south velocity. This is the case even if the nonaveraged north-south velocity can be considered as in geostrophic balance with the east-west gradient of pressure (or height) field.
- (c) In the eddy forcing terms for the mean circulation equatorial wave and gravity waves, which are important components of the forcing, are filtered out in the geostrophic version of the theory.
- (d) The constraint of geostrophy is so strong that one is prevented from addressing some fundamental problems concerning zonally averaged circulations using the set of geostrophic equations. For example, it is probably not appropriate in using the set of geostrophic equations to deduce the state of the atmosphere in the absence of zonal asymmetry, because while these equations would lead one to conclude that the symmetric atmosphere is extremely close to radiative equilibrium,

while the original primitive equations suggest otherwise.

A major hindrance in the past in adopting the more general (i.e. non-geostrophic) formulation of zonally averaged circulations lies not in technical difficulties associated with the formulation but in clear interpretation of the role of various eddy forcing terms. This problem is largely circumvented when the zonal averaging is performed on isentropic surfaces, as it is done in the present work. The role of eddies in the forcing of the mean flow is much better defined in isentropic coordinates without the presence in isobaric coordinates of mean temperature advection by the ageostrophic circulation.

The more general formulation suggests that a hypothetical atmosphere can be in different circulation regimes depending on the strength of the eddy Eliassen-Palm flux divergence. When the eddy strength is strong, the mean circulation may be in the geostrophic regime away from the equator. However, when the eddy strength is weak, the circulation may be close to the “nearly inviscid nonlinear regime” of Held and Hou (1980), which is not a continuation of the geostrophic regime as the eddy strength is reduced.

In Part II, we will address the question of which regime *our* atmosphere is in. Furthermore, we will use our present isentropic formulation and the derived relation between Eliassen-Palm flux pseudo divergence and the flux of Ertel’s isentropic potential vorticity to deduce the isentropic diffusion coefficient,  $K_{yy}$  that is useful in tracer transport studies.

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## Appendix A.

**The zonally averaged momentum equation in isentropic coordinates on a sphere.**

Starting from the zonal momentum equation:

$$\begin{aligned} \frac{\partial}{\partial t} u + \frac{u}{a \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{v}{a} \frac{\partial}{\partial \varphi} u + \dot{\theta} \frac{\partial}{\partial \theta} u \\ - \left( 2\Omega + \frac{u}{a \cos \varphi} \right) \sin \varphi v = - \frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda} \Phi \end{aligned} \quad (A.1)$$

and the equation for the conservation of mass:

$$\frac{\partial}{\partial t} \rho_{\theta} + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda} \rho_{\theta} u + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (\rho_{\theta} v \cos \varphi) + \frac{\partial}{\partial \theta} (\rho_{\theta} \dot{\theta}) = 0, \quad (A.2)$$

one has

$$\begin{aligned} \frac{\partial}{\partial t} \rho_{\theta} u + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda} \rho_{\theta} u u + \frac{1}{a \cos^2 \varphi} \frac{\partial}{\partial \varphi} (\rho_{\theta} u v \cos^2 \varphi) + \frac{\partial}{\partial \theta} (\rho_{\theta} u \dot{\theta}) \\ - 2\Omega \sin \varphi \rho_{\theta} v = - \frac{\rho_{\theta}}{a \cos \varphi} \frac{\partial}{\partial \lambda} \Phi \end{aligned} \quad (A.3)$$

Taking the zonal average:  $(\bar{\cdot}) \equiv \int_0^{2\pi} (\cdot) d\lambda$ , one obtains the zonally averaged momentum equation:

$$\begin{aligned} \frac{\partial}{\partial t} \overline{\rho_\theta u} + \frac{1}{a \cos^2 \varphi} \frac{\partial}{\partial \varphi} (\overline{\rho_\theta u v} \cos^2 \varphi) + \frac{\partial}{\partial \theta} (\overline{\rho_\theta u \dot{\theta}}) - 2\Omega \sin \varphi \overline{\rho_\theta v} \\ = - \overline{\frac{\rho_\theta}{a \cos \varphi} \frac{\partial}{\partial \lambda} \Phi} \end{aligned} \quad (A.4)$$

We let, as in Tung (1982):

$$\bar{U} \equiv \overline{\rho_\theta u}, \bar{V} \equiv \overline{\rho_\theta v} \cos \varphi, \text{ and } \bar{W} \equiv \overline{\rho_\theta \dot{\theta}} \quad (A.5)$$

be the zonally averaged mass circulation, and define, as in Gallimore and Johnson (1981a) the density-weighted zonal average

$$\hat{u} \equiv \overline{\rho_\theta u} / \bar{\rho}_\theta = \bar{U} / \bar{\rho}_\theta,$$

and deviation from zonal average

$$u^* \equiv u - \hat{u}. \quad (A.6)$$

Then

$$\begin{aligned} \overline{\rho_\theta u v \cos^2 \varphi} &= \overline{\rho_\theta [\hat{u} + u^*] \left[ \frac{\bar{V}}{\bar{\rho}_\theta} + v^* \cos \varphi \right] \cos \varphi} \\ &= \hat{u} \bar{V} \cos \varphi + \overline{\rho_\theta u^* v^* \cos^2 \varphi} + \frac{1}{\bar{\rho}_\theta} \overline{\rho_\theta u^* \cos \varphi} \bar{V} + \hat{u} \cos^2 \varphi \overline{\rho_\theta v^*} \end{aligned}$$

The last two terms are zero from definition (A.6). Eq. (A.4) now becomes

$$\begin{aligned} \frac{\partial}{\partial t} \bar{\rho}_\theta \hat{u} + \frac{1}{a \cos^2 \varphi} \frac{\partial}{\partial \varphi} (\hat{u} \cos \varphi \bar{V}) + \frac{\partial}{\partial \theta} (\hat{u} \bar{W}) - 2\Omega \sin \varphi \bar{V} / \cos \varphi \\ = - \overline{\left( \frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda} \Phi \right) \rho_\theta} - \frac{1}{a \cos^2 \varphi} \frac{\partial}{\partial \varphi} (\overline{\rho_\theta u^* v^* \cos^2 \varphi}) - \frac{\partial}{\partial \theta} (\overline{\rho_\theta u^* \dot{\theta}}) \end{aligned} \quad (A.7)$$



The right hand side of (A.7) can be written as

$$\frac{1}{\cos \varphi} \vec{\nabla} \cdot \vec{F} = \frac{1}{\cos \varphi} \left[ \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} F_y + \frac{\partial}{\partial \theta} F_\theta \right]$$

where

$$F_y \equiv -\overline{\rho_\theta u^* v^*} \cos^2 \varphi,$$

and

$$F_\theta \equiv - \left[ \overline{\rho_\theta u^* \dot{\theta}^*} - \frac{1}{g} \overline{p \frac{\partial}{a \cos \varphi \partial \lambda} \Phi} \right]. \quad (\text{A.8})$$

In order to write the first term on the right-hand side of (A.7) into a vertical derivative term, we have made use of the hydrostatic equation

$$g \rho_\theta = - \frac{\partial}{\partial \theta} p$$

and the relationship between the Montgomery streamfunction  $\Phi$  and pressure

$$\frac{\partial}{\partial \theta} \Phi = c_p \left( \frac{p}{p_{00}} \right)^{R/c_p}.$$

Finally, by multiplying both sides of Eq. (A.7) by  $\cos \varphi$ , we obtain

$$\frac{\partial}{\partial t} (\bar{\rho}_\theta \hat{u} \cos \varphi) + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (\hat{u} \cos \varphi \bar{V}) + \frac{\partial}{\partial \theta} (\hat{u} \cos \varphi \bar{W}) - 2\Omega \sin \varphi \bar{V} = \vec{\nabla} \cdot \vec{F} \quad (\text{A.9})$$

Next, the mean momentum equation expressed in terms  $\bar{u}$  (instead of  $\hat{u}$  as in Eq. (A.9) is derived. We start with Eq. (A.1) and perform the zonal average,

$$\frac{\partial}{\partial t} (\bar{u} \cos \varphi) + \overline{\frac{v}{a} \frac{\partial}{\partial \varphi} u \cos \varphi} + \overline{\dot{\theta} \frac{\partial}{\partial \theta} u \cos \varphi} - 2\Omega \sin \varphi \bar{v} \cos \varphi = 0 \quad (\text{A.10})$$

We let  $v = \hat{v} + v^*$ , but  $u = \bar{u} + u'$ , then

$$\overline{\frac{v}{a} \frac{\partial}{\partial \varphi} u \cos \varphi} = \frac{\hat{v}}{a} \frac{\partial}{\partial \varphi} \bar{u} \cos \varphi + \overline{\frac{v^*}{a} \frac{\partial}{\partial \varphi} u' \cos \varphi} + \frac{\overline{v^*}}{a} \frac{\partial}{\partial \varphi} \bar{u} \cos \varphi$$

Since

$$v^* = v' - \overline{\rho'_\theta v'} / \bar{\rho}_\theta$$

from the definition of  $v^*$ , and so

$$\overline{a^* b'} = \overline{a' b'},$$

we have

$$\overline{\frac{v}{a} \frac{\partial}{\partial \varphi} u \cos \varphi} = \frac{\hat{v}}{a} \frac{\partial}{\partial \varphi} \bar{u} \cos \varphi + \overline{\frac{v'}{a} \frac{\partial}{\partial \varphi} u' \cos \varphi} - \frac{1}{\bar{\rho}_\theta} \overline{\rho'_\theta v'} \frac{1}{a} \frac{\partial}{\partial \varphi} \bar{u} \cos \varphi.$$

Similarly,

$$\overline{\hat{\theta} \frac{\partial}{\partial \theta} u \cos \varphi} = \hat{\theta} \frac{\partial}{\partial \theta} \bar{u} \cos \varphi + \overline{\hat{\theta}' \frac{\partial}{\partial \theta} u' \cos \varphi} - \frac{1}{\bar{\rho}_\theta} \overline{\rho'_\theta \hat{\theta}'} \frac{\partial}{\partial \theta} \bar{u} \cos \varphi.$$

Substituting

$$\bar{v} = \hat{v} - \overline{\rho'_\theta v'} / \bar{\rho}_\theta$$

into (A.10), we find

$$\begin{aligned} & \frac{\partial}{\partial t} (\bar{u} \cos \varphi) + \frac{\hat{v}}{a} \frac{\partial}{\partial \varphi} \bar{u} \cos \varphi + \hat{\theta} \frac{\partial}{\partial \theta} \bar{u} \cos \varphi - 2\Omega \sin \varphi \hat{v} \cos \varphi \\ & = -\frac{\overline{v'}}{a} \frac{\partial}{\partial \varphi} u' \cos \varphi - \overline{\hat{\theta}' \frac{\partial}{\partial \theta} u' \cos \varphi} \\ & - \left( 2\Omega \sin \varphi - \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \bar{u} \cos \varphi \right) \overline{\rho'_\theta v'} \cos \varphi / \bar{\rho}_\theta + \frac{1}{\bar{\rho}_\theta} \overline{\rho'_\theta \hat{\theta}'} \frac{\partial}{\partial \theta} \bar{u} \cos \varphi \quad (A.11) \end{aligned}$$

Using the zonally averaged continuity equation (from (A.2)):

$$\frac{\partial}{\partial t} \bar{\rho}_\theta + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (\bar{\rho}_\theta \hat{v} \cos \varphi) + \frac{\partial}{\partial \theta} (\bar{\rho}_\theta \hat{\theta}) = 0 \quad (A.12)$$

we rewrite (A.11) finally into the form

$$\begin{aligned} & \frac{\partial}{\partial t} (\bar{\rho}_\theta \bar{u} \cos \varphi) + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (\bar{\rho}_\theta \hat{v} \cos \varphi \bar{u} \cos \varphi) \\ & + \frac{\partial}{\partial \theta} (\bar{\rho}_\theta \hat{\theta} \bar{u} \cos \varphi) - 2\Omega \sin \varphi \hat{v} \cos \varphi = \square \cdot \mathcal{F}, \end{aligned} \quad (\text{A.13})$$

where

$$\begin{aligned} & \frac{1}{\bar{\rho}_\theta} \square \cdot \mathcal{F} \equiv -\overline{\frac{v'}{a} \frac{\partial}{\partial \varphi} u' \cos \varphi} - \overline{\dot{\theta}' \frac{\partial}{\partial \theta} u' \cos \varphi} \\ & - \left( 2\Omega \sin \varphi - \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \bar{u} \cos \varphi \right) \overline{\rho'_\theta v' \cos \varphi / \bar{\rho}_\theta} + \frac{1}{\bar{\rho}_\theta} \overline{\rho'_\theta \dot{\theta}'} \frac{\partial}{\partial \theta} \bar{u} \cos \varphi \end{aligned} \quad (\text{A.14})$$

## References

- Andrews, D. G. and M. E. McIntyre, 1978a: Generalized Eliassen-Palm and Charney-Drazin theorems for waves on axisymmetric mean flows in compressible atmospheres. *J. Atmos. Sci.*, **35**, 2031-2048.
- Andrews, D. G. and M. E. McIntyre, 1978b: On wave-action and its relatives. *J. Fluid Mech.*, **89**, 647-664.
- Andrews, D. G., 1983: A finite-amplitude Eliassen-Palm theorem in isentropic coordinates. *J. Atmos. Sci.*, **40**, 1877-1883.
- Andrews, D. G., J. D. Mahlman and R. W. Sinclair, 1983: Eliassen-Palm diagnostics off wave; mean-flow interaction in the GFDL "SKYHI" general circulation model, *J. Atmos. Sci.*, **40**, 2768-2784.
- Andrews, D. G., and M. E. McIntyre, 1976: Planetary waves in horizontal and vertical shear: the generalized Eliassen-Palm relation and the mean zonal acceleration. *J. Atmos. Sci.*, **33**, 2031-2048.
- Batchelor, G. K., 1956: On steady laminar flow with closed streamlines at large Reynolds number. *J. Fluid Mech.*, **1**, 177-190.
- Benney, D. J. and D. R. S. Ko, 1978: The propagation of long large amplitude internal waves, *Stud. Appl. Math.*, **59**, 187-199.
- Boyd, J., 1976: The noninteraction of waves with the zonally averaged flow on a spherical earth and the interrelationships of eddy fluxes of energy, heat and momentum. *J. Atmos. Sci.*, **33**, 2285-2291.
- Bretherton, F. P., 1966: Critical layer instability in baroclinic flows. *Quart.*

*J. Roy. Meteor. Soc.*, **92**, 325-334.

Charney, J. G., and M. E. Stern, 1962: On the instability of internal baroclinic jets in a rotating atmosphere. *J. Atmos. Sci.*, **19**, 159-172.

Clough, S. A., N. S. Grahame and A. O'Neil, 1985: Potential vorticity in the stratosphere derived using data from satellites *Quart. J. Roy. Meteor. Soc.*, **111**, 335-358.

Dickinson, R. E., 1969: Theory of planetary wave-zonal flow interaction. *J. Atmos. Sci.*, **26**, 73-81.

Dunkerton, T. J., C-P. F. Hsu and M. E. McIntyre, 1981: Some Eulerian and Lagrangian diagnostics for a model sudden warming. *J. Atmos. Sci.*, **38**, 819-843.

Edmon, H. J., B. J. Hoskins and M. E. McIntyre, 1980: Eliassen-Palm cross sections for the troposphere. *J. Atmos. Sci.*, **37**, 2600-2616. [See also Corrigendum, 1981: *J. Atmos. Sci.*, **38**, 115.]

Eliassen, A., and E. Palm, 1961: On the transfer of energy in stationary mountain waves. *Geofys. Publ.*, **22**, No. 3, 1-23.

Ertel, H., 1942: Ein neuer hydrodynamischer Wirbelsatz. *Meteorol. Z.*, **59**, 271-281.

Gallimore, R. G., and D. R. Johnson, 1981a: The forcing of the meridional circulation of the isentropic zonally averaged circumpolar vortex. *J. Atmos. Sci.*, **38**, 583-599.

Gallimore, R. G., and D. R. Johnson, 1981b: A numerical diagnostic model of

- the zonally averaged circulation in isentropic coordinates. *J. Atmos. Sci.*, **38**, 1870-1890.
- Green, J. S. A., 1970: Transfer properties of the large-scale eddies and the general circulation of the atmosphere. *Quart. J. Roy. Meteor.*, **96**, 157-185.
- Hartmann, D. L., 1977: On potential vorticity and transport in the stratosphere. *J. Atmos. Sci.*, **34**, 968-977.
- Held, I. M., A. Y. Hou, 1980: Nonlinear axially symmetric circulations in a nearly inviscid atmosphere. *J. Atmos. Sci.*, **37**, 515-533.
- Hide, R., 1969: Dynamics of the atmospheres of the major planets with an appendix on the viscous boundary layer at the rigid boundary surface of an electrically conducting rotating fluid in the presence of magnetic field. *J. Atmos. Sci.*, **26**, 841-853.
- Holton, J. R. 1979: An Introduction to Dynamic Meteorology. Second edition, Academic Press, 391 pages.
- Holton, J. R., and W. M. Wehrbein, 1980: A numerical model of the zonal mean circulation of the middle atmosphere. *Pure Appl. Geophys.*, **118**, 284-306.
- Hoskins, B. J., M. E. McIntyre, and A. W. Robertson, 1985: On the use and significance of isentropic potential vorticity maps. *Quart. J. Roy. Met. Soc.*, **111**, 877-946.
- Hoskins, B. J., and F. P. Bretherton, 1972: Atmospheric frontogenesis mod-

- Palmer, T. N., 1982: Properties of the Eliassen-Palm flux for planetary scale motions. *J. Atmos. Sci.*, **39**, 992-997.
- Pedlosky, J. 1979: *Geophysical Fluid Dynamics*. Springer-Verlag, 624 pages.
- Reed, R. J. and K. E. German, 1965: A contribution to the problem of stratospheric diffusion by large-scale mixing. *Monthly Wea. Rev.*, **93**, 313-321.
- Schneider, E. K., 1977: Axially symmetric steady-state models of the basic state for instability and climate studies. Part II: Nonlinear calculations *J. Atmos. Sci.*, **34**, 280-297.
- Schneider, E. K., and R. S. Lindzen, 1977: Axially symmetric steady-state models of the basic state for instability and climate studies. Part I: Linearized calculations. *J. Atmos. Sci.*, **34**, 263-279.
- Stern, M. E., 1975 : Minimal properties of planetary eddies. *J. Mar. Res.*, **33**, 1-13.
- Taylor, G. I., 1915: Eddy motion in the atmosphere. *Phil. Trans. Roy. Soc. London*, **A215**, 1-26.
- Tung, K. K., 1982: On the two-dimensional transport of stratospheric trace gases in isentropic coordinates. *J. Atmos. Sci.*, **39**, 2330-2355.
- Tung, K. K., 1984: Modeling of tracer transport in the middle atmosphere, in *Dynamics of the Middle Atmosphere*, J. R. Holton and T. Matsuno editors, 417-444.
- Tung, K. K., T. F. Chan and T. Kubota, 1982: Large amplitude internal

waves of permanent form. *Stud. Appl. Math.*, **66**, 1-44.

Wehrbein, W. M. and C. B. Leovy, 1982: An accurate radiative heating and cooling algorithm for use in a dynamical model of the middle atmosphere. *J. Atmos. Sci.*, **39**, 1532-1544.



- els: mathematical formulation and solution. *J. Atmos. Sci.*, **29**, 11-37.
- Houghton, J. T., 1978: The stratosphere and mesosphere. *Quart. J. Roy. Met. Soc.*, **104**, 1-29.
- Leovy, C., 1964: Simple models of thermally driven mesospheric circulation. *J. Atmos. Sci.*, **21**, 327-341.
- Leovy, C. B., C.-R. Sun, M. H. Hitchman, E. E. Remsberg, J. M. Russell III, L. L. Gordley, J. C. Gille and L. V. Lyjak, 1985: Transport of ozone in the middle stratosphere: evidence for planetary wave breaking. *J. Atmos. Sci.*, to appear.
- Mahlman, J. D., D. G. Andrews, D. L. Hartmann, T. Matsuno and R. G. Murgatroyd, 1984: Transport of trace constituents in the stratosphere in *Dynamics of the Middle Atmosphere*, p. 387-416. J. R. Halton and T. Matsuno, editors. Terra Scientific Publishing Company, Tokyo, Japan.
- McIntyre, M. E., and T. N. Palmer, 1983: Breaking planetary waves in the stratosphere. *Nature*, **305**, 13, 593-600.
- McIntyre, M. E., and T. N. Palmer, 1984: The 'surf zone' in the stratosphere. *J. Atmos. Terr. Phys.*, **46** 825-849.
- Nakamura, H., 1978: Dynamical effects of mountains on the general circulation of the atmosphere. III. Effects on the general circulation of the baroclinic atmosphere. *J. Meteor. Soc. Japan*, **56**, 353-366.